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POTENTIAL

POTENTIAL

AND

ITS APPLICATION TO THE EXPLANATION OF ELECTRICAL PHENOMENA

POPULARLY TREATED

BY

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WATERLOO PLACE, LONDON

MDCCCLXXXIX

TRANSLATOR'S PREFACE

THE translation of Dr. O. Tumlirz's book on potential has been undertaken from a strong conviction that it will prove useful to students of physics.

The term potential was originally introduced into the language of science in order to denote an abstract mathematical conception, which, when introduced physically, is seen to afford a simple and comprehensive expression for some of the most fundamental relations involving the idea of force. The notion of potential has thus come to form the starting-point of the modern theoretical treatment of several very important branches of science, and notably of electricity and magnetism; familiarity with it has consequently become essential for the accurate study of even the elementary parts of these subjects.

In consequence, however, of the essentially mathematical nature of the conception, and of the fact that it was introduced into science from the

mathematical and not from the physical side, an adequate and systematic exposition of the idea of potential has hitherto been confined almost entirely to books written from a somewhat advanced mathematical point of view, in which the matter has been presented in too abstract a manner, and often with too much of mathematical technicalities to be accessible to elementary students.

In the present volume Dr. Tumlirz has set himself to treat the same conception from the physical side, and, while all but avoiding the use of technical mathematics, has succeeded in producing a work which not only affords to non-mathematical readers a fuller exposition of the subject than is accessible to them in existing English treatises, but one which may be read with profit by students who are already familiar with the purely mathematical aspect of the same subject.

There does not appear to be any elementary, systematic treatise on this subject in the English language; and in elementary text-books on physics, potential can necessarily only be briefly explained. The book will, it is hoped, supplement the usual elementary text-books on electricity and magnetism, and give students a firmer grasp of the physical meaning of potential than is

possible to be obtained from a less extended exposition.

The book contains a considerable amount of information on collateral subjects not usually found in ordinary text-books.

The translator is very greatly indebted to Professor Carey Foster, F.R.S., of University College, London.

Professor Foster has very kindly revised the proofs throughout, and has assisted in more ways than it is possible to enumerate. Numerous improvements in the terminology originally adopted are due to the same source.

LONDON, *September* 1889.

PREFACE

COMPREHENSIVE classification of acquired knowledge is the chief object in every science, and this object will be accomplished by seeking for such points amongst known facts as really indicate all groups of the same kind, and as it were draw red threads through them. If we observe the phenomena of heavy masses, of electricity and magnetism, we find in the case of each the same law of force. We might have in this law a guiding principle of the phenomena named; it appears, however, that it is not so much this force, but rather a certain work of this force, the potential, that affords in a preferable manner the desired view of the facts.

In the text-books which treat of the theory of potential this is at first introduced as a mathematical expression and treated throughout as such, whilst the treatment merely takes the form of discussing the properties of this expression, of this function. This may well be the reason why such vague views often prevail regarding the physical meaning of potential and its properties—why so often in the study of the phenomena of electricity some confound potential with density, others potential with electrical force. It is for this reason that the author of the present work, whose real object is to make a wider circle acquainted with the properties of potential, starts at first with the physical meaning

of potential, and in deducing the properties of potential keeps steadily in view their physical side.

The object of the book demands as few suppositions as possible. The notions, force, mass, density, work, and so on, will therefore be subjected to a preliminary discussion; and the mathematical apparatus, consisting only in the use of elementary mathematics, will everywhere be limited to a minimum.

The book is divided into four chapters: the first treats of the potential of gravity, the second and third of electrical potential in its application to statical electricity and galvanic currents, and the fourth of magnetic, electromagnetic, and electro-dynamic potential.

THE AUTHOR.

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ERRATA

Page 74.—After end of sentence in line 7 from top, *read:* "The electricity which is found on the opposite part of the conductor is named influence-electricity of the second kind."

Page 139.—In heading of Section 41, *for* "conductor". *read* "condenser."

INTRODUCTION

AUXILIARY PROPOSITIONS FROM MECHANICS

SECTION 1.—**Uniform Motion.**

IF a material point, *i.e.* an infinitely small particle (atom, molecule) of which the extension may be entirely disregarded, moves, it describes in the course of time a certain line or curve which we name its path. If we keep this motion in view from a definite point of the path, and assign for each period of time, t, the corresponding distance passed over, s, we may enter the results in a table, putting the periods of time in one column, and the corresponding spaces in another. If we can now substitute for this table a rule for its construction, *i.e.* if we can find an equation which contains the universal relation between space and time, we can by this, if the path is known, assign the position of the moving particle for each point of time, *i.e.* we know the motion. That equation is named the *law of the distances on the path.*

The material particle moves, for instance, on a straight line according to the law

$$s = a + bt,$$

where a and b are two constant (unchanging) magnitudes. The direction of this motion will be always

the same, viz. that of the straight line, and the distances described in equal times will always be equal to each other. In each unit of time, *i.e.* in every second, the same distance will be described. This distance is called the *velocity* of the motion. The direction and velocity are the same at every instant: the motion therefore takes place in the same manner; it is, as it is commonly said, uniform. We can graphically represent the characteristics of this motion as follows:

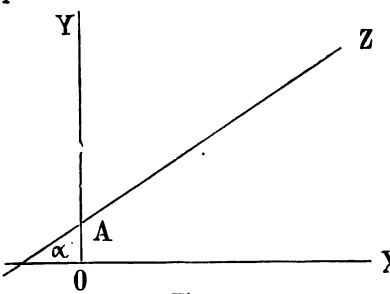

Fig. 1.

We draw (Fig. 1) the straight lines OX and OY at right angles to one another. We mark off along OX from the point O spaces to represent the times. We further draw from the terminal points of these spaces rectilinear distances parallel to OY, and make these equal to the spaces corresponding to the times. The terminal points of the last spaces give, when joined together, a straight line AZ, which makes an angle α with the straight line OX. The straight line AZ is the geometrical representation of the equation $s = a + bt$. The constant b is equal to tan α, or the velocity is represented by the tangent of the angle of incidence α.

SECTION 2.—**Variable Motion.**

If motion takes place so that the direction remains the same but the velocity changes, or in such a way that the velocity remains the same but the direction

changes, or in such a manner that both change, so that the motion is different at different times, it is then variable motion.

We will consider first the case in which the direction remains the same but the velocity changes. Such a motion takes place in a straight line, and if the velocity increases, the motion is said to be accelerated, and if the velocity decreases, it is said to be retarded. The retardation is considered negative acceleration ; and we designate the rate of change of velocity, *i.e.* the amount by which the velocity changes in the unit of time (second) as the *acceleration.*

We will consider a particular case of rectilinear variable motion, viz. that for which the law of the distances has the form

$$s = a + bt + ct^2,$$

where a, b, c signify constant (unchanging) quantities. We may write this equation in the form

$$s = a + (b + ct)t,$$

and if we compare it with the equation

$$s = a + bt$$

given above, which held for uniform motion and in which b denoted the constant velocity of the motion, we then see that we may consider the motion between two infinitely near points of time at the time t as uniform motion, which takes place with the velocity $b + ct$. In other words, the velocity has at the time t the value $b + ct$. If c is a positive quantity, then the velocity will continually increase ; if, on the other hand, c is negative, the velocity will continually decrease. c is the amount by which the speed changes in the unit of time, or the acceleration. When the change is constant, *i.e.* is always the same both in magnitude and direction, the motion is called uni-

formly accelerated motion. An example of uniformly accelerated motion is afforded by the free fall of a body whose increase of velocity per unit of time, or acceleration at the equator and at the sea level, is 9.78009 metres.

If in another case the velocity is always the same in amount but the direction changes, the motion, as has been already stated, is a variable one. That an acceleration also takes place here may be shown in a simple manner. Let a material point move in a straight line *mn* (Fig. 2) with the uniform velocity *b*. On arriving at *n* let it change the direction of its motion and move from *n* towards *o*, again uniformly, with the same velocity *b* and in the time τ. It would also have arrived at the same point *o* if it had moved along *np* in the original direction with the same velocity, and having arrived at *p* had then taken the course *po*. If we construct the parallelogram *npoq*, we can say: If a second velocity had united itself to the original velocity *b* at the point *n*, which had the direction *nq*, and in consequence of which the point, if it possessed this alone, would, in the time τ have arrived at *q*, it would under the influence of both of these velocities traverse the space *no* in the time τ with the velocity *b*. The deflection of the direction of motion has indeed the same meaning as if there were added to the original velocity a new one in the direction *nq*, *i.e.* the material point receives when it has arrived at the point *n* an increase of velocity, or an acceleration in the direction *nq*. For example, if a material point moves in a circle of radius *r* and describes in each second the distance *v*, its motion,* although it describes equal distances in

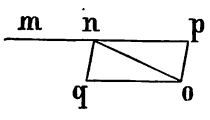

Fig. 2.

* This motion is frequently designated "uniform curvilinear."

equal times, is still attended with an acceleration which possesses the magnitude $\frac{v^2}{r}$ and is directed towards the centre. It bears the name of centripetal acceleration. When v and r have the same value at all points of the path the acceleration has also everywhere the same magnitude.

Circular motion affords an analogy with the uniformly accelerated motion described above. In the case of both the acceleration is constant; but whilst the acceleration in the latter case acts in the direction of motion, the velocity being in equal times equally increased, the direction however not being changed, in the former case the acceleration is at right angles to the motion, and the motion continually changes its direction, viz. in equal times by an equal amount, but the velocity remains unchanged.

That in the case of variable motion, whose magnitude and direction both continually change at every instant, an acceleration must occur, follows immediately from the consideration of the above mentioned simple cases.

SECTION 3.—**Force and Mass.**

The material bodies which are found in the external world are in themselves quite passive. If such a body is found in a condition of rest it does not of itself change this condition; and conversely, if it is found in a condition of motion it continues in this condition, if no external force acts on it, unchanged both as regards direction and velocity; its motion goes on uniformly. A leading principle of mechanics is expressed in this statement, the so-called *principle of inertia.* The external causes determining motion are commonly named *forces.* If a body passes from a state of rest to a state

of motion, or if it abandons its uniform motion, or, in other words, if it experience a change of velocity,* external forces must act on it. We therefore see that the effect of forces consists in the production of acceleration. As this statement follows from the principle of inertia, so conversely the principle of inertia can be deduced from the statement. Galilei, the founder of dynamics, or the doctrine of the motion of material bodies, was the first to perceive that forces produce acceleration, and from this he deduced the principle of inertia.†

The force is put proportional to the acceleration, *i.e.* if the acceleration is 2, 3, 4 times as great, we infer that the force is 2, 3, 4 times as great. If we denote the acceleration by γ and the force by P we can represent it in the usual way,

$$P = m\gamma,$$

where m is a constant factor. Experience shows that one and the same force acting on different bodies produces entirely different accelerations, or that one and the same body acquires from different forces different accelerations, or that different bodies which experience the same acceleration are acted on by different forces. If I take, for example, a stone in my hand, I can throw it a certain distance by the exertion of a certain force. If I take a second stone of the same structure, but of greater size, I cannot throw it so far by exerting the same force; I must employ a greater force in order to throw it as far as the first stone. Otherwise all bodies acquire the same acceleration from gravity, but if I raise different bodies from the ground, different amounts of force must be exerted by me. We observe therefore that the

* The condition of rest may be regarded as a case of motion with velocity zero.

† Compare Mach, *Die Mechanik in ihrer Entwicklung historisch und kritisch dargestellt.* Leipzig: Brockhaus, 1883, § 131.

earth must attract different bodies with different forces in order to impart to them equal accelerations. The quantity $m = P/\gamma$ is different for different bodies. We can express this in the usual manner by saying that different bodies exhibit different amounts of vis inertiæ; the quantity m, the measure of this vis inertiæ, is termed the *mass* of the body. Let us consider a definite force, viz. the force of the attraction of the earth, or the so-called force of gravitation. The force with which a body is attracted by the earth is called its weight. All bodies under the influence of gravity acquire the same acceleration. We will denote this by g, the weight of a body by Q, and its mass by M, and there results, in accordance with the above,

$$Q = Mg \text{ or } M = \frac{Q}{g},$$

i.e. the *mass of a body is proportional to the weight.* The weight Q can be expressed by a number if it is compared with the unit weight; the quantity of mass can also be represented numerically. The weight of a cubic centimetre of pure distilled water at 4° C. may be taken as the unit weight. The mass of the unit of weight would accordingly be expressed by $\frac{1}{g}$. If the masses of all other bodies are compared with this mass, definite proportionate numbers are obtained. It is usually by these numbers that the masses of bodies are expressed. The numbers express the ratio of the mass of a body to that of a cubic centimetre of pure distilled water at 4° C. Since the masses are proportional to the weights, the ratio of the masses is the same as the ratio of the weights.

The mass contained in the unit volume, *i.e.* in a cubic centimetre at the temperature of T degrees, is named the *density* of the body at the temperature T.

We can express the density by a number if we compare it with the mass of a cubic centimetre of pure distilled water, and take this last mass as the unit. Since the masses are proportional to the weights, the ratio of the masses in question will be equal to the ratio of their weights. This latter ratio bears the name "specific weight." Density and specific weight are as proportionate numbers equal to one another.

SECTION 4.—**The Parallelogram of Forces.**

Let two forces act on a material particle A (Fig. 3) in the direction AB and AC. The one would by itself produce the acceleration AB, the other the acceleration AC. The material particle can, however, only experience an acceleration in one definite direction. This acceleration will be found if we construct a parallelogram on AB and AC, and draw the diagonal. The diagonal AD represents the resulting acceleration of the material particle both in amount and direction. The effect of the forces acting is precisely the same as if only one force had acted in the direction AD. The last force, which replaces both the given forces, is called their resultant, and the forces its components. This is expressed by saying, "The two components AB and AC are compounded into the resultant AD by the parallelogram ABCD." The parallelogram ABCD is called the parallelogram of forces. In the same manner as we can compound two forces into one we can deal with three, four, or more forces. The converse problem is to resolve one force into two other forces. This problem is only soluble if either the direction of both forces is

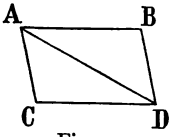

Fig. 3.

given, or one is given both in direction and magnitude. If, for example, the force AD is given, and we wish to resolve this into two components parallel to the directions AB and AC. We draw from D the straight lines BD and CD parallel to the given directions, and we get in the lengths AB and AC the components required. Or, if it is required to resolve the force AD into two components, one of which, AB, is given in magnitude and direction, we join D and B, then draw from A and D the straight lines AC and CD parallel to the straight lines BD and AB, we then get in AC the second component required. We can resolve these again into components, and so on.

SECTION 5.—**Work of a Force.**

Let the material particle A be acted on by a force AC (Fig. 4), and let it describe under the influence of this force the rectilinear distance AB. We will assume that the force during the motion is constant in magnitude and direction. If α is the angle which the force makes with the direction of displacement, the work of the force AC during the displacement AB is $AC \times AB \times \cos \alpha$. If we let fall from the point B the perpendicular BD on the direction AC, then the distance AD, the projection of the distance AB on the direction AC, is equal to $AB \times \cos \alpha$, and we can therefore represent the work by $AC \times AD$; or in words, *the work of the force* AC *during the displacement* AB *is equal to the product of the magnitude of the force and the projection of the displacement on the direction of the force.*

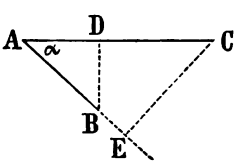

Fig. 4.

If we let fall from the point C the perpendicular CE on the direction of displacement, then the distance AE, the projection of the force AC on the direction of displacement, is equal to AC × cos α. We can therefore express the work by AB × AE, or in words, *the work of the force* AC *during the displacement* AB *is equal to the product of the magnitude of the displacement and the projection of the force on the direction of displacement.* If the direction of displacement coincides with the direction of the force, then $\alpha = 0$, and the work is therefore equal to the product of the force and the displacement.

If the force is at right angles to the direction of displacement, or if $\alpha = 90°$, then cos $\alpha = 0$, and the work

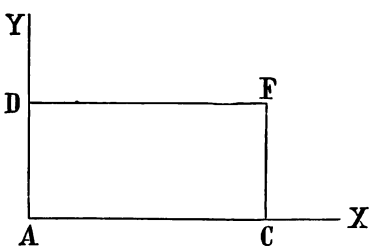

Fig. 5.

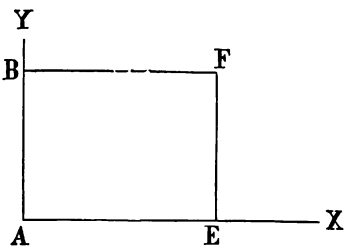

Fig. 6.

is therefore equal to zero. The force has then no influence on the displacement.

If the direction of the force forms with the direction of displacement an obtuse angle, or if α is greater than 90°, then the cosine of α is negative, and consequently the work is negative. The force in this case has a tendency to hinder the motion of the particle.

We have at one time represented the work by AC × AD, at another by AB × AE; we see by this that work can be represented by an area. If we draw two straight lines, AX, AY, at right angles to one another, and mark off (Fig. 5) on the straight line AX the force

AC, and on the straight line AY the distance AD, *i.e.* the projection of the displacement on the direction of the force, then the area of the rectangle ACFD represents the amount of the work done. We can also (Fig. 6) mark off on the straight line AX the distance AE, *i.e.* the projection of the force on the direction of displacement, and on the straight line AY the displacement AB, and upon these construct the rectangle AEFB. This area again represents the work done.

We may now suppose that the displacement occurs always in the same direction, in the same straight line, but that the force continuing in the same direction changes its magnitude as time goes on, or that the magnitude continuing the same the direction changes, or that both change. Then the projection of the force on the direction of displacement will be different from time to time. How can we now represent the work? Let OX (Fig. 7) be the direction of displacement. We erect at every position of the moving point a perpendicular which for this time is exactly equal to the projection of the force on the direction of displacement, and then join the terminal points of these distances; in this way we obtain a certain bent line or curve. Suppose that the moving point has arrived at M, and let MM′ be the projection of the force corresponding to this time. After a short time it arrives at N, then NN′ will be the corresponding projection. Since MM′ differs very little from NN′, we can make no great mistake by assuming that for the displacement from M to N the projection is represented by MM′, or that the

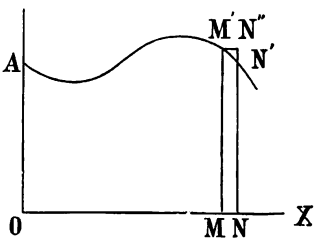

Fig. 7.

work done will be represented by the area of the rectangle MM′N″N. This area differs only very little from the area MM′N′N, and this difference will be less in proportion as the distance MN is smaller ; the difference will almost vanish if we suppose MN to be very small. We can thus break up the area OAM′M into a number of such small strips ; each represents an amount of work, viz. that work which would be done by the displacement along the base ; consequently the area OAM′M represents the total work done owing to the displacement from O to M.

From the proposition that the work of a force during a rectilinear displacement is equal to the product of the displacement and the projection of the force on the displacement, it follows that if we resolve one force into two or more components, *the work of the resultant is equal to the sum of the works of the components.* In Fig. 8 AC is the resultant, AB and AD are the components. The work of the resultant during displacement through the distance AF is

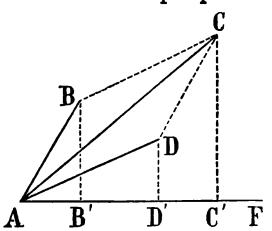

Fig. 8.

$$AC' \times AF,$$

the work of the component AB is

$$AB' \times AF,$$

and that of the component AD

$$AD' \times AF.$$

The sum of the works of both components is

$$AF(AB' + AD').$$

Now, however, since $BC = AD$, $AD' = B'C'$, therefore $AB' + AD' = AB' + B'C' = AC'$, and consequently the sum of the works of the components is equal

$$AF \times AC',$$

i.e. equal to the work of the resultant. If we resolve the component AB into two components, we obtain three components, and if we also resolve AD into two components we obtain in all four components, and so on. By the application of the proposition proved for two components, it immediately follows that the sum of the works of all components, however great their number may be, is always equal to the work of the resultant.

If a material particle moves in space on a curved line, and if at the same time the force changes both in magnitude and direction, we can determine the work in the following manner. We resolve at every moment the force into three components, OX, OY, OZ (Fig. 9), in directions at right angles to each other, and determine the work of each one of these. According to the above, the sum of these works is equal to the work of the resultant. In order to determine the work of one component, we must consider that the work of a force during a small displacement is equal to the product of the force and the projection of this displacement on the direction of the force. We shall obtain, for example, the work of the component acting in the direction OX if at each instant we project the displacement that takes place in the next following element of time on the direction OX, and multiply by this the magnitude of the component. The very same relations result, as we have considered, in Fig. 7. We can therefore represent the work of each component by an area, and consequently the total work by three areas.

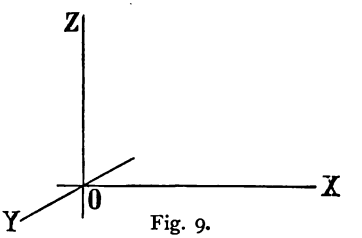

Fig. 9.

CHAPTER I

POTENTIAL OF GRAVITY

SECTION 6.—**Equipotential Surfaces and Lines of Force of the Earth.**

WE will now apply the propositions arrived at in the preceding part to a particular case. Let a heavy body fall vertically downwards so that its centre of gravity describes the space AB or s (Fig. 10). We name the force which draws it down its weight; let this be denoted by Q. When this force is constant both in direction and magnitude, and is directed vertically downwards, the amount of the work is equal to the product Qs.

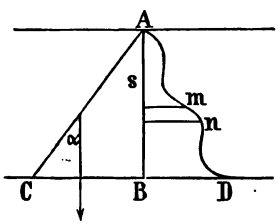

Fig. 10.

Now we may suppose that the body falls down an inclined plane AC, and that its centre of gravity describes the path AC and arrives at the horizontal plane which passes through B. The force forms with the displacement an angle, viz. the angle α; the work is consequently expressed by the product $Q.AC \cos \alpha$, or, since $AC \cos \alpha = s$, by Qs. The work is still the same as in the first case. Indeed the work of gravity

remains always the same, viz. Qs, whatever path the centre of gravity of the body describes from the point A to the horizontal plane BC. Let us for example consider the path AD.

We shall obtain the work which will be performed in the very short time in which the centre of gravity passes from m to n, if we project the distance mn on the direction of the force or on AB, and multiply this by Q. This gives $Q \cdot mn \cdot \cos(AB \cdot mn)$. We can imagine the distance AD broken up into a great many such portions as mn; if we determine the work done for each, the total of these will be expressed by their sum, or, if we denote the sum by Σ, by $\Sigma Q \cdot mn \cdot \cos(AB \cdot mn)$.

In every term of the sum the quantity Q occurs; if this is taken out $Q \cdot \Sigma\, mn \cos(AB \cdot mn)$ is obtained, *i.e.* Q multiplied by the sum of the projections of the divisions mn on the straight line AB, or as this sum is the distance AB itself, consequently the magnitude again is expressed by the product Qs.

If a horizontal plane is drawn through the point A, *and a moving particle be caused to start from any point whatever of the plane, the force of gravity, when the moving body has arrived at the plane* BC, *will always have performed the same amount of work, viz. the work* Qs, *whatever may have been the direction of the moving body.*

Both the planes considered are at right angles to the direction of gravity; they are, strictly speaking, not planes, but parts of very great surfaces which possess nearly the form of spheres. If we then imagine both these planes extended so as to be always at right angles to the direction of gravity, we obtain two closed surfaces which surround the earth, and possess nearly the form

of spheres. These surfaces are called level surfaces. With regard to the expression level surfaces (Niveauflächen), it is derived from the fact that a liquid in a vessel arranges itself in such a manner that the surface, the level, extends horizontally, *i.e.* at right angles to the direction of gravity, and in all points of a plane which is parallel to the level the hydrostatic pressure is the same. The level is only apparently a plane ; it is in reality a part of an immensely large closed surface which surrounds the earth, and extends everywhere at right angles to the direction of gravity, and possesses nearly the form of a sphere, as is very well observed at the surface of the sea. If the whole earth were covered with water, the surface of the water would represent a level surface of the kind described.

We can illustrate the level surfaces of the earth by a further example. Our earth's atmosphere forms a spherical shell. If we ascend vertically from the surface of the earth, the pressure of the air will become always smaller and smaller. If we imagine to ourselves all points of the atmosphere which have the same pressure joined together, we obtain very nearly concentric spherical surfaces whose centre lies in the centre of the earth. We can imagine an infinite number of such nearly concentric spherical surfaces constructed, which everywhere extend at right angles to the direction of gravity, and we can also represent to ourselves that the space situated beyond the earth is bounded by such a surface of infinitely great radius. In future, when we speak of infinitely distant points, we will always understand by this, points in this infinitely large equipotential surface.

The equipotential surfaces of the earth are nearly spherical surfaces ; for the sake of simplicity we will

assume that they are exactly spherical surfaces. As such they are all concentric and successively envelop each other. A material particle on any one of these surfaces will move, if left to the influence of gravity, towards an inner surface. The work which gravity performs, if it attracts the material particle from a more distant to a nearer surface, is always the same whatever the path may be which is described.

The direction of gravity is very nearly that of the straight lines running out from the centre. If a material particle follow quite freely the force of attraction of the earth, its path nearly coincides with one of these straight lines. Assuming, for the sake of simplicity, that the equipotential surfaces are exactly spherical, we may also assume that the direction of gravity corresponds entirely with those straight lines. The straight lines to which the equipotential surfaces are at right angles are called lines of force, viz. lines of force of the earth.

SECTION 7.—**Potential of the Earth.**

If a material particle, at which we imagine the unit of mass concentrated, comes from one equipotential surface to a second, the work performed by gravity is independent of the direction. We now imagine that the material particle is situated at an infinite distance, and moves from thence, under the influence of gravity, towards the earth, and arrives at a position through which the equipotential surface, which we will denote by N, passes. Then the work of gravity is independent of the direction of motion, and always the same at whatever point of the observed equipotential surface the material particle may arrive.

If we wish to ascertain the amount of the work, we

can simplify the matter by assuming that the path of the moving particle is always a straight line which is determined by the given point and the centre of the earth. During such motion, the direction of the force everywhere coincides with the displacement. Let us imagine O (Fig. 11) to be the centre of the earth, and OA to be its radius. If we erect at each point of the straight line OX, at right angles to this, rectilinear distances which represent by their lengths the forces of gravity acting on the unit of mass at these points, and

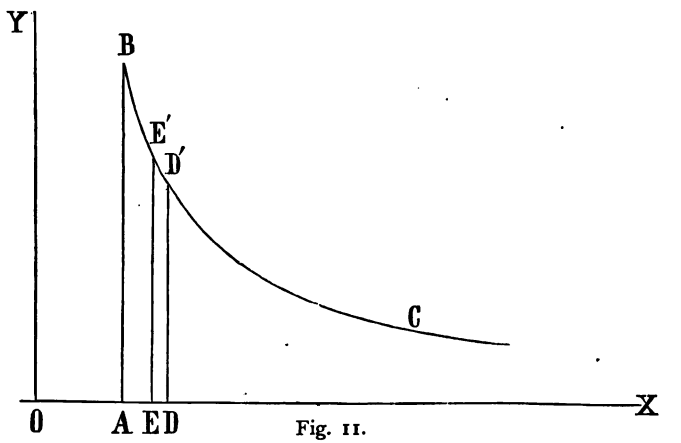

Fig. 11.

join the terminal points of all these distances with one another, we obtain a curve BC, which approaches the axis OX asymptotically. The area which is enclosed by the axis OX, the ordinate AB, and the curve BC, represents the amount of work which the force of gravity performs by the approach of the unit of mass from infinity to the surface of the earth. This work, as has already been mentioned, is always the same whatever path the unit of mass may have described, and at what-

ever point of the earth's surface it may have arrived. If the unit of mass does not arrive so far as the earth's surface, but only at the equipotential surface N, so that the distance from the centre of the earth equals OD, then the work done is expressed by that area which is bounded by the straight lines DX and DD′ and the curve D′C. There corresponds, as we see, to each different distance from the centre of the earth a different amount of work ; or in other words, *if the unit of mass is on a definite surface there corresponds to this a definite amount of work on the part of the force of gravity.* The nearer the equipotential surface lies to the earth's surface, the greater the work, and on the earth's surface it is greatest. At all points in one and the same surface we obtain the same amount of work ; *the equipotential surfaces are therefore surfaces of equal work.*

The work which gravity performs if the unit of mass supposed to be concentrated at a point moves from infinity to a point situated at a finite distance, is called *the gravitation potential at this point.** Since for all points of the level surface which we can draw through this point the work has the same value, the level surfaces are surfaces of equal potential.

During the motion of the material particle under consideration from an infinite distance towards the surface of the earth the work increases continuously, viz. in such a way that the work during the motion of the material particle for an infinitely small distance also increases by an infinitely small amount, *i.e.* the work increases continuously, or, in other words, the potential increases continuously.

The potential has in all points of space a definite

* The name potential was first introduced by the English mathematician Green.

value. The difference of the value of potential at two different points is called the potential difference of these points.

SECTION 8.—The Relation of Potential to the Force of Attraction.

Let us imagine that the unit of mass is situated at the distance OD from the centre of the earth. The surface between DX, DD', and the curve D'C represents, as we have seen, the total work done on the part of the earth on the unit of mass. If the unit of mass now moves under the influence of gravity towards the earth, viz. in the direction of the line of force, which goes through its position (Fig. 12), and it thereby comes to an inner equipotential surface situated infinitely near the first, to E, then work will be done during this motion. The work is represented by the portion of surface DED'E' (Fig. 11). This portion of the surface we can approximately consider a rectangle, which is erected on the base DE. This base, moreover, signifies simply the displacement of the unit of mass. If we divide the area of the rectangle by the base we obtain the force acting in the direction of displacement, or, since the displacement has taken place along the line of force, the acting force itself.

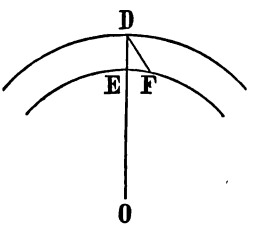

Fig. 12.

If the material particle is displaced from D to F (Fig. 12), then, because F and E are on the same equipotential surface, exactly the same amount of work is done. We can resolve the acting force into two components—into one which coincides with the direction

of displacement and one at right angles to this. The work of the latter is equal to zero. The former component alone does the work mentioned. We shall obtain this component if we divide the work done by the displacement. We arrive then at the following result: If the unit of mass comes from one equipotential surface to a second infinitely near it, the total work done on it, or the potential, increases by a definite amount. *We obtain the force component acting in the direction of displacement if we divide the increase of potential by the amount of displacement*, or, in other words, *the potential difference of two very near points, divided by their distance apart, gives the component of force acting in the line joining the points.* If the material particle is moved along an equipotential surface, no increase of potential occurs, and hence the components of force, falling in the direction of displacement, are equal to zero, which, moreover, follows immediately from the definition of equipotential surfaces.

During the motion of the material particle the force changes continuously, *i.e.* it changes by an infinitely small amount if the material particle passes over an infinitely small space.

SECTION 9.—**Newton's Law of Gravitation.**

Sir Isaac Newton first inferred, from the influence of the earth on the motion of the moon, that gravity imparts acceleration not only to bodies on the earth's surface, but also throughout the universe. He also recognised the converse, viz. that forces proceed from heavenly bodies which, acting on our earth, produce acceleration; the grand phenomenon of the ebb and flow of the tides is a beautiful example of the effects

of such forces. The property of gravity belongs not only to our earth, but to all masses moving in space. Since it is not local, does not belong individually to the earth, its seat is not confined to the earth's centre, as was formerly supposed. Every portion, however small, has its own share, and every portion produces acceleration on every other.*

If we suppose two material particles, A and B, at a distance r, then A will be attracted by B and B by A; in both cases the force of attraction is the same. Newton first showed that this force diminishes with the distance, that it is 4, 9, 16 . . . times as small if the distance is 2, 3, 4 . . . times as great, or that the force varies inversely as the square of the distance of the material particle.

If we name the mass of the particle A, m, and that of B, m', then the force which acts on the particle A is proportional to $\frac{mm'}{r^2}$ or equal $k \cdot \frac{mm'}{r^2}$, where k denotes a proportional factor. In what follows, for the sake of simplicity, the quantity k will always have the value 1.

In this, Newton's law of attraction (gravitation), the supreme law of the motion of heavenly bodies, is expressed, and this has been confirmed by experiment and observation in a most beautiful manner.†

We can formulate it in the following way :—

Two material particles exert on one another an attractive force. The direction of this force is that of the straight line joining the particles. The magnitude of the force is directly proportional to the product of the two masses, and inversely proportional to the square of their distance.

* Cf. Mach, *loc. cit.* p. 178.

† The discovery of Neptune by Adams and Leverrier is a beautiful verification of the universality of Newton's law of gravitation.—TR.

SECTION 10.—**Potential of a Homogeneous Spherical Shell.**

Let us imagine the earth to have a completely spherical form, and let it be throughout of the same density, or homogeneous. Let us further imagine the attracted unit of mass to enter into the earth through its surface and arrive at P (Fig. 13). If we describe round the point O with radius OP a spherical surface, we may consider the attraction which the material particle experiences at P to be compounded of the attraction of the inner sphere POS and the spherical shell SAB. In order to determine the attraction of the spherical shell we consider first the attraction of a very thin spherical shell on a particle M situated in its interior, and at which we imagine the unit of mass to be concentrated (Figs. 14 and 15).

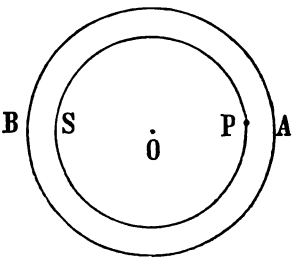

Fig. 13.

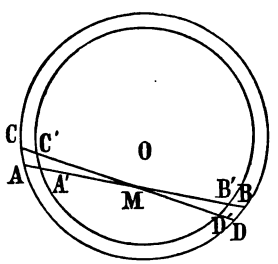

Fig. 14.

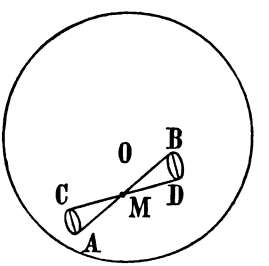

Fig. 15.

We draw (Fig. 14) the straight lines AB and CD. If we join the points AC and DB by straight lines, we obtain two similar triangles, namely, the triangle AMC

and the triangle DMB. From this similarity the proportion AC : DB = CM : BM follows.

If the arcs AC and DB are very small, they will coincide with the straight lines AC and BD, and we can therefore substitute in the last proportion the arcs AC and DB for the straight lines AC and BD.

We obtain in precisely the same way the proportion A'C' : B'D' = C'M : B'M. It is further to be observed that AA' = BB' and CC' = DD'.

We can, as regards CMA and BMD, represent them as two opposite cones whose vertices are situated in M and whose vertical angle is the angle AMC. These cones cut out from the outer spherical surface (Fig. 15) two portions of surface AC and BD, which are related to each other as AC^2 to BD^2, or as CM^2 to BM^2. The volumes, therefore, cut out from the spherical shell by the cones, and consequently the masses contained in these volumes, are related to each other as CM^2 to BM^2. The forces of attraction, moreover, which these masses exert on the unit of mass at M are related, according to Newton's law of gravitation, as

$$\frac{\text{mass ACC'A'}}{CM^2} : \frac{\text{mass BB'D'D}}{BM^2},$$

or as

$$\frac{\text{mass ACC'A'}}{\text{mass BB'D'D}} : \frac{CM^2}{BM^2},$$

or, by reason of the former proportion, as

$$\frac{CM^2}{BM^2} : \frac{CM^2}{BM^2},$$

or as

$$1 : 1,$$

i.e. the two forces of attraction are equal to one another. Since they are also in opposite directions, they will neutralise one another at the point M.

This, moreover, holds wherever the double cone

ACMDB may be situated. If we give to it all possible positions, then we can draw the whole spherical shell within the compass of our investigation. Since the forces will always be neutralised, the total force of attraction of the spherical shell on the particle M will be zero, or *the spherical shell exerts on the unit of mass, when brought into the hollow space in its interior, no force of attraction.* If the unit of mass is displaced while within the hollow space, the forces of attraction do no work. The circumstances are just the same as if the point were displaced on an equipotential surface.

Let us investigate for this case the value of the potential. If the unit of mass is situated outside the spherical shell, it will be attracted by a force which is directed towards the centre, because in reality the mass of the spherical shell is distributed in a completely symmetrical manner about the straight line joining the unit of mass with the centre. *The lines of force of the spherical shell are therefore straight lines which run out from the centre, consequently the equipotential surfaces are spherical surfaces which are concentric with each other and with the spherical shell.* The nearer the unit of mass approaches the spherical shell the smaller will be its distance from each portion of mass of the spherical shell, and the greater therefore will be the force of attraction. As the unit of mass moves from infinity towards the spherical shell, the work done by the attracting force, or the potential, continually increases. If it penetrate the very thin spherical shell and arrive at the interior, the total work, or potential, does not then change, since the force of attraction now vanishes. *The potential has not only everywhere the same value at all points in the interior hollow space, but also the greatest value which in this case it can possibly attain.*

From what has been said, it follows that the attraction of the spherical shell SAB in Fig. 13 on the unit mass concentrated at the point P is equal to zero. For we can consider this spherical shell as made up of exceedingly numerous, extremely thin concentric spherical shells, all of which enclose the point P and therefore exert no force on it. It is thus only the sphere POS that acts on the material particle P.

The equipotential surfaces of this sphere are concentric spherical surfaces with the centre at O, the lines of force are straight lines proceeding from O. Since the point P may be chosen anywhere, we see that the equipotential surfaces of the homogeneous sphere are also concentric spherical surfaces in the interior of the same. The attracting force which the unit mass experiences at the point P is naturally less than at the point A; it will continually be smaller the nearer P approaches to O, and will disappear completely at the point O. Since the force only disappears at this point, the potential must continually increase during the motion of the unit of mass towards the point O, and at the point O must attain its greatest value.

SECTION 11.—Attraction between two Homogeneous Spheres—Potential of the Spheres on one another.

We will now consider the case of two spheres. We will consider two homogeneous spheres of equal mass and volume, and therefore of equal density. We will for the sake of brevity designate them "sphere I" and "sphere II" (Fig. 16). Let us consider first sphere I. The equipotential surfaces of this are concentric spherical surfaces with the centre at A. If the unit of mass

arrives from infinity at a place situated at a finite distance, then the work of the force of attraction of sphere I is the potential due to this sphere at that position.

If, however, in place of the unit of mass the mass m comes from infinity to the given point, the work done by sphere I is m times the former amount, or equal to the potential multiplied by m. We may now suppose sphere II to be entirely broken up into small pieces, and imagine to ourselves that these portions of mass were originally infinitely distant, and that they have arrived at their present position from infinity and have here united so as to form sphere II. The attracting force of sphere I will have done work on all these portions of mass. We shall obtain the amount of this work if we multiply the separate portions of mass by the value of the potential which is produced by sphere I at the positions where the portions of mass occur, and then find the sum of all these products. The sum gives us what is called the potential of sphere I on sphere II.

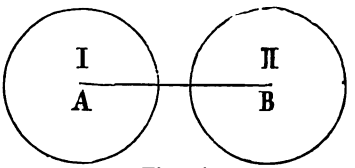

Fig. 16.

This potential, as a matter of course, is independent of the direction which the separate portions of mass of sphere II have described in the course of their motion. We can for the sake of simplicity assume that the portions have already united to form sphere II, while at an infinite distance, and that these, as a whole, have arrived at their present position in such a way that the centre B has always moved along the straight line AB.

If we now fix our attention on sphere II, and reason in the same way as before, we can obtain the

potential of sphere II on sphere I by multiplying the separate portions of mass of sphere I by the potential value which will be produced by sphere II at the corresponding positions, and summing all these products. This potential is equal to the work which the force of attraction of sphere II performs, if, sphere II remaining fixed, we bring sphere I from infinity into its present position in such a way that the centre always remains on the straight line AB. The potential of sphere II on sphere I is equal to the potential of sphere I on sphere II. According to the principle of the equality of action and reaction which we owe to Newton, sphere II will be attracted by sphere I with the same force as sphere I by sphere II. If in the first case sphere II has moved a certain distance from sphere I, it will be attracted by this with a definite force in the direction BA. If in the second case sphere I has moved this same distance from sphere II, it will be attracted by this with the same force in the direction AB. The work must, therefore, in both cases be the same.

SECTION 12.—Potential of a Homogeneous Sphere on itself.

Let us now consider sphere I for the present by itself. We imagine the whole mass divided into small portions, and represent to ourselves that these portions were originally all at an infinitely great distance from one another, and by their reciprocal attractions were brought to their present positions. Work will, as a matter of course, be done. We obtain this if we consider successively the various portions, taken two at a time, and determine the work which is done by their

motion from an infinite distance apart to their present relative position, and add the amounts of work thus obtained. We will denote the entire work by P, and will now undertake the task of giving a simple expression for this with the help of the potential of the sphere. For this purpose let us imagine all the portions of mass of the sphere to be doubled, but in such a way that the double mass occupies the same space as before, and therefore that at all points double density prevails. Then, it is evident that the total work which is performed by the motion of these portions of mass from an infinite relative distance to their present positions will be fourfold that in the former case, or 4P, because the masses are doubled and the force of the mutual attraction of two masses is proportional to their product. I can imagine the present sphere of twofold density produced by sphere I having been penetrated by an identically equal sphere, or I can imagine the present sphere made up of two spheres, I′ and I″, coinciding with one another, each of which is exactly equal to sphere I. We now imagine to ourselves every portion of sphere I″ brought from its present position to infinity until finally only sphere I′ remains, then work will be consumed by overcoming the force of attraction. Since the work P inheres in sphere I′ remaining behind, the work consumed amounts to 3P. This consists of two parts ; the one part refers to the force of attraction of sphere I′, and will be obtained if we multiply every portion of mass of sphere I″, while it is united with sphere I′, by that value of potential which sphere I′ produces at the respective positions, and sum these products ; let this work be denoted by P_1. The other part of the work refers to the mutual attraction of the portions of mass of sphere I″, and is evidently equal to

the work which is performed if these portions of mass, coming from an infinite distance, unite to form sphere I″, or, since sphere I″ is identical with sphere I, it is equal to P. We accordingly obtain the equation—

$$3P = P_1 + P,$$

or

$$P = P\tfrac{1}{2_1},$$

i.e. *the work which is inherent in the sphere, and which will be done if its portions of mass coming from infinitely great distances apart unite to form the sphere, will be obtained, if every portion of the mass of the sphere is multiplied by the value of the potential prevailing there, and the sum of these products added and divided by* 2. *This work* P *is named "the potential of the sphere on itself."*

There is also inherent in sphere II a similar amount of work ; it is, viz., " the potential of sphere II on itself." Since sphere II is exactly equal to sphere I, the potential of sphere II on itself is equal to the potential of sphere I on itself. The total work which is present in the system of both the spheres I and II is made up of the potential of sphere I on itself, the potential of sphere II on itself, and the potential of sphere I on sphere II.

If the potential of sphere I on sphere II, which, as we know, is equal to the potential of sphere II on sphere I, be considered as divided into two equal parts, viz. into half the potential of sphere I on sphere II and into half the potential of sphere II on sphere I, and if we further consider the definition given for the potential of one sphere on itself, we can, if we still call the sum of the potentials of both spheres at a given point the potential of the system at this point, express the total work considered by saying, the total work will be found if we multiply every portion of mass of the system by half the potential of the system prevailing there and add all these products together. The total work of

the system of both spheres will therefore be found in the same manner as the potential of a sphere on itself: this total work is called also *the potential of the system on itself, or the energy or work of the system.*

SECTION 13.—Potential of two distant Homogeneous Spherical Shells on one another.

We can carry on our investigations in a precisely similar manner if we have to do with two homogeneous spherical shells. (Fig. 16 may be used for illustration of this case.)

If we suppose spherical shell II divided into separate portions of mass, and each of these multiplied by the value of potential which the spherical shell I produces at the corresponding position, and all these products summed, we then obtain the potential of spherical shell I on spherical shell II. This is equal to the work which is done if we hold spherical shell I fast and allow spherical shell II to move from infinity to its present position.

If we divide spherical shell I into separate portions of mass and multiply each of these by the potential value which spherical shell II produces at the corresponding position, and sum these products, we obtain the potential of spherical shell II on spherical shell I. This is again equal to the work which is done if we hold fast spherical shell II and allow spherical shell I to move from infinity to its present position.

We can prove that these amounts of work are equal to one another, according to the principle of the equality of action and reaction, in precisely the same way as we did formerly in the case of spheres; we need, for instance, merely replace the word "sphere" by "spherical shell."

SECTION 14.—The Relation between a Homogeneous Spherical Shell and a portion of Mass concentrated at its centre.

Let us now consider a very thin homogeneous spherical shell and a portion of mass M (Fig. 17) concentrated at its centre.

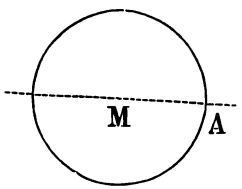

Fig. 17.

The lines of force of the mass M are straight lines which run out from M; consequently the equipotential surfaces of the mass M are concentric spherical surfaces, with their centre at M.

Let us now imagine that the whole mass of the spherical shell has moved to infinity, just as, for example, a spherical undulation extends, then work will be consumed in overcoming the attracting force of the mass M, and, in fact, so that this work is equal to that which the attracting force of M performs, if the mass of the spherical shell coming from infinity became united so as to form the actual spherical shell. This work is nothing else than the potential of the mass M on the spherical shell. The spherical shell itself forms an equipotential surface of M. If a mass arrives from infinity on this surface, the work is always the same on whatever part of the surface it has arrived, and conversely, if a mass goes from the surface to infinity, the work consumed will be the same, from whatever point of the surface it starts. The work which is consumed in overcoming the force of attraction of M, by the transference of the entire mass of the spherical shell from this to infinity, is not changed, if we suppose all the divisions into which we have imagined the spherical shell to be broken up to be brought along the spherical

shell to the point A, and there united into a material point, and then imagine this to be moved along the straight line MA into infinity. The work done will in reality be the same if the material point, which contains the mass of the spherical shell, comes from infinity to the point A. The potential of the material point M on the spherical shell is equal to this work. If the material particle M comes from infinity to the centre of the spherical shell, then the work done by the attracting force of the spherical shell is equal to the potential of the spherical shell on the mass M. This work is the sum of the amounts of work which the attracting forces of the individual portions of mass of the spherical shell perform. The equipotential surfaces of one of those portions of mass are concentric spherical surfaces. If a material point comes from infinity to such a surface, then the work done by the attracting force of the portion of mass is the same on whatever point of the surface it has come, or it is solely determined by the final distance apart. Now the mass M has the same distance from all portions of the mass of the spherical shell; the work, therefore, done by the attracting force of the spherical shell during the motion of the mass M from infinity to the centre is the same as if the whole mass of the spherical shell were concentrated at the point A. We can therefore say, the potential of the spherical shell on the mass M is the work which is performed by the attracting force of the spherical shell, if we suppose its mass concentrated at the point A and the mass M (for the sake of simplicity) brought to M from infinity along the straight line MA. The potential of the mass M on the spherical shell was, however, the work which the attracting force of the mass M performed when the mass of the spherical shell,

considered as concentrated at a point, moved from infinity to A. It follows, therefore, again from the principle of the equality of action and reaction, that the amounts of the work of both are equal to one another, or that *the potential of the spherical shell on the material particle* M *is equal to the potential of the material particle* M *on the spherical shell.*

SECTION 15.—Two Concentric Homogeneous Spherical Shells.

Of two concentric homogeneous spherical shells, which we will denote by I and II (Fig. 18), let the former contain the mass m, the latter the mass m', and let the spherical shell II enclose the spherical shell I.

If we now imagine the spherical shell II divided into separate portions, and that these have been brought from infinity to their present positions, the work done by the attracting force of I during the motion will be the potential of spherical shell I on spherical shell II. From previous considerations we know that the equipotential surfaces of a homogeneous spherical shell are concentric spherical surfaces, which have their centre common with the spherical shell. Spherical shell II will therefore be an equipotential surface of spherical shell I, and accordingly the work which is performed by the attraction of a portion of mass of II during its motion from infinity is exactly the same on whatever part of the spherical surface II it has arrived. The total work which is performed by the attraction of all the portions of mass of spherical shell II is therefore

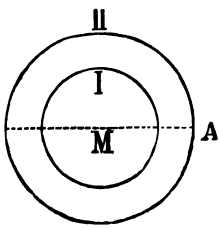

Fig. 18.

the same as if all the portions of mass, by their motion from infinity, had arrived at the point A of spherical shell II and had become concentrated there. Since the work is independent of the paths traversed by the various portions of the mass, we can imagine these already concentrated at a point at infinity, and that they (for the sake of simplicity) have moved to A along the straight line MA under the influence of the attracting force of sphere I. This work is, as has been said, equal to the potential of spherical shell I on spherical shell II.

In order to find the potential of spherical shell II on spherical shell I, we must take into consideration that the potential of spherical shell II has the same value at all points of the inner hollow space, because here the attractive force of the spherical shell is zero. If the portions of mass of spherical shell I come from infinity to their present position, then the work which the attracting forces of spherical shell II have performed will no longer be altered if we bring all the portions to the centre M and concentrate the mass of spherical shell I at the centre. The potential of spherical shell II on spherical shell I is therefore equal to the potential of a spherical shell on a material particle of mass m situated at its centre, or, as we saw, equal to the work that will be done if we concentrate the mass of spherical shell II at a point, A, on itself, and imagine the mass of spherical shell I, concentrated at a point, brought to the point M from infinity (from left to right) (for the sake of simplicity) along the straight line MA.

We can still express this potential in another manner: The work of the attracting forces of spherical shell II is made up of the quantities of work of the

attracting forces of the separate portions of mass. The work which one such portion of mass performs, by attracting the separate portions of spherical shell I, is determined by the final relative position of this portion as regards that portion of mass. Now, however, the final relative position of all portions of I to every portion of II will be the same, and consequently the quantities of work which the attracting forces of the separate portions of mass of spherical shell II perform will also be the same, if we imagine the portions of mass of spherical shell II concentrated at a point A on spherical shell II, and bring spherical shell I from infinity to its present position. This work is the potential of spherical shell II on spherical shell I. In accordance with the principle of the equality of action and reaction, it immediately follows that this potential is equal to the potential of spherical shell I on spherical shell II. But still more follows from this consideration. We have, on the one hand, found the potential of spherical shell II on spherical shell I equal to that work which will be done if we hold fast the mass of spherical shell II at A and bring the total mass of spherical shell I, considered as concentrated at a point, from infinity (for the sake of simplicity) along the straight line MA to M; and on the other hand, we saw that this work is equal to that which the attracting force of the mass of spherical shell II, considered as concentrated at a point A, performs, if we move spherical shell I from infinity (for the sake of simplicity) in such a way towards A that the centre always remains on the straight line MA and arrives at M. By the principle of the equality of action and reaction, the amounts of work resulting must be equal. We, in the first place, hold the mass of spherical shell I in a fixed position at the point M,

supposed to be concentrated there, and imagine the mass of spherical shell II, supposed to be concentrated at A, to have come to A from infinity. Secondly, we keep spherical shell I in a fixed position, so that its centre remains at M, and again imagine the mass of spherical shell II, considered as concentrated at the point A, to have come from infinity to A. Both quantities of work must be exactly equal to one another. It follows from this that *the spherical shell* I *acts outwardly, just as the material particle* M, *in which the whole mass of the material shell is imagined to be concentrated, or, in other words, it acts at a distance just as if its whole mass were concentrated at the centre.*

SECTION 16.—The Magnitude of the Force of Attraction of a Homogeneous Spherical Shell and a Homogeneous Sphere.

If the result last obtained is compared with Newton's law of gravitation, then the magnitude of the attracting force of a homogeneous spherical shell is given at once. According to Newton's law of attraction, the force for instance with which a material particle of mass m attracts another of mass n, at the distance r, is equal $\frac{mn}{r^2}$. *The force, therefore, with which a homogeneous spherical shell of mass* m *attracts a material particle of mass* n *at the distance* r *from the centre of the spherical cell is equal* $\frac{mn}{r^2}$.

We can consider a homogeneous sphere to be composed of exceedingly numerous very thin spherical shells. The force with which such a sphere attracts a material particle situated outside its surface is equal to

the sum of the forces of attraction of the spherical shells. Since every spherical shell acts as if its mass were concentrated at the centre, so the whole sphere will act outwardly just as if its whole mass were concentrated at the centre. If the mass of the whole sphere is M, and the mass of the material particle n, and the distance of this from the centre of the sphere is r, then the attracting force of the sphere is equal

$$\frac{Mn}{r^2}.$$

Moreover, in the case when the sphere is not homogeneous, but the density changes in such a manner that it possesses the same value at all points which are equally distant from the centre, the amount of the attracting force is immediately given. We can in fact imagine the sphere to be made up of very numerous extremely thin homogeneous spherical shells, which possess different densities. Since each of these spherical shells acts outwardly as if its mass were concentrated in the centre, so also will the whole sphere act in the distance as if its whole mass were concentrated in the centre, or its attractive force is equal to

$$\frac{Mn}{r^2},$$

if we denote by M the whole mass of the sphere, by n the mass of the material particle, and by r the distance of this from the centre.

SECTION 17.—**Potential due to a Material Particle.**

Let the mass M be concentrated at the point O (Fig. 19), and let this attract the unit mass which we likewise imagine to be concentrated at a point. If the unit mass can freely follow the attraction, it moves on one of the

lines of force of the mass M, *i.e.* on the straight line passing through O.

Let the potential at the point A have the value A, and at the point Z the value Z. During the motion of the unit mass from A to Z the attractive force of the mass M has done the work Z — A. Now we know from what precedes that if a constant force F displaces a material particle in the direction of the force through a distance a, then the work amounts to F. a. In our case the force indeed acts in the direction of displacement, but, as we have seen from Newton's law of attraction, it

Fig. 19.

changes its amount continuously. In order to be able to deduce the amount of work from the law of force, we imagine the distance AZ broken up into very small divisions such that we can consider the force during its motion through such a division to be constant. If we multiply each division by the magnitude of the corresponding force, the sum of these products gives us the potential difference Z — A.*

Let us now assume the mass M to have the value unity, and let the distance of the points A, B, C, ... Z from the point O be $a, b, c, \ldots z$. The force with which the unit mass at O acts on the unit mass at A has the magnitude $\frac{1}{a^2}$, at B $\frac{1}{b^2}$, at C $\frac{1}{c^2}$, ... at Z $\frac{1}{z^2}$ and so on. All these forces act in the direction AO. $\frac{1}{a^2}$ is the smallest, $\frac{1}{b^2}$ the greatest force on the distance AB. If we multiply the distance AB or $a - b$ by $\frac{1}{a^2}$, this product is smaller

* Cf. Clerk-Maxwell, *Elementary Treatise on Electricity*, Art. 86.

than the actual work done; if, on the other hand, we multiply $a-b$ by $\frac{1}{b^2}$, this product is greater than the actual work done.

If we denote the potential at the points A, B, C, ... Z by A, B, C, ... Z, we may write

$$\frac{a-b}{a^2} < B-A < \frac{a-b}{b^2},$$

or

$$\left(\frac{1}{b}-\frac{1}{a}\right)\frac{b}{a} < B-A < \left(\frac{1}{b}-\frac{1}{a}\right)\frac{a}{b}.$$

We obtain an entirely similar equation for the space BC, namely,

$$\left(\frac{1}{c}-\frac{1}{b}\right)\frac{c}{b} < C-B < \left(\frac{1}{c}-\frac{1}{b}\right)\frac{b}{c},$$

and so on.

The ratios $\frac{a}{b}, \frac{b}{c}$, etc. are all greater than 1.

Let us assume that the greatest value of these ratios is p. The ratios $\frac{b}{a}, \frac{c}{b}$ and so on are merely the reciprocal values of the former ratios, consequently they are all less than 1, and the least equals $\frac{1}{p}$. We therefore obtain—

$$\left(\frac{1}{b}-\frac{1}{a}\right)\frac{1}{p} < B-A < \left(\frac{1}{b}-\frac{1}{a}\right)p$$

$$\left(\frac{1}{c}-\frac{1}{b}\right)\frac{1}{p} < C-B < \left(\frac{1}{c}-\frac{1}{b}\right)p$$

. . . . .

$$\left(\frac{1}{z}-\frac{1}{y}\right)\frac{1}{p} < Z-Y < \left(\frac{1}{z}-\frac{1}{y}\right)p.$$

If we add all these inequalities we find

$$\left(\frac{1}{z}-\frac{1}{a}\right)\frac{1}{p} < Z-A < \left(\frac{1}{z}-\frac{1}{a}\right)p.$$

If the number of points between A and Z is increased, the divisions considered will become smaller and smaller, a condition which brings the magnitude p and its reciprocal value $\frac{1}{p}$ always nearer to unity. And if the number of the divisions into which the distance AZ has been divided becomes exceedingly great, then p and $\frac{1}{p}$ can be considered as units, and by this we arrive at the result

$$Z-A=\frac{1}{z}-\frac{1}{a}.$$

In order to deduce from this the value of the potential at Z, we must consider that the potential at Z signifies the work which the attracting force performs if the material particle which contains the unit mass moves from infinity to Z. We assume that the point A is situated at infinity and is infinitely distant from Z. The potential A is then equal to zero; for the force which proceeds from the unit mass at O has no value at an infinite distance. In whatever manner the unit mass is moved at an infinite distance, the work of the force proceeding from O is always equal to zero, so long as it remains infinitely distant from O. Now, since $\frac{1}{a}$, when a is infinitely great, is equal to zero, then the potential Z is expressed by

$$Z=\frac{1}{z},$$

or in words: *The numerical value of the potential at a given point, produced by the unit mass at a given distance,*

is equal to the value of the reciprocal of the number which expresses this distance.

If the mass M is concentrated at the point O, then the potential at the distance z is equal to

$$\frac{M}{z}.$$

If we have to deal with a number of material particles which are situated at different distances from a given point, we shall obtain the potential at this point if we add the potentials which the material particles singly produce.

SECTION 18.—Potential of a Homogeneous Spherical Shell and a Homogeneous Sphere.

In section 15 we arrived at the result that a homogeneous spherical shell acts at a distance just as if its whole mass were concentrated at its centre.

We have, with the help of this proposition and Newton's law of attraction, determined in section 16 the magnitude of the attractive force of a homogeneous spherical shell and of a homogeneous sphere. We now can, with the help of the potential of a material particle, obtained in the last paragraph, also express the potential of a homogeneous spherical shell and a homogeneous sphere. If m is the mass of the former, then the potential due to this at an external point at the distance r is equal to

$$\frac{m}{r}.$$

At the points on the surface it possesses the value $\frac{m}{a}$, when a denotes the radius and retains this value—the spherical shell being assumed to be very thin—at all points of the interior hollow space.

A homogeneous sphere can, as already stated, be considered to be composed of numberless concentric homogeneous spherical shells. If M therefore denotes the mass of the sphere, then the potential at an external point at a distance r from the centre is equal to

$$\frac{M}{r},$$

and at the points on the surface equal to

$$\frac{M}{a},$$

when a denotes the radius of the sphere.

In the case when the sphere is not homogeneous, precisely the same expression is obtained if the density changes in such a way that at all points equally distant from the centre it possesses the same value, so that the sphere can be supposed to be made up of exceedingly numerous homogeneous spherical shells of different densities.

SECTION 19.—**Force and Mass (Density) continued.**

A material particle, in which the mass m is concentrated, attracts the unit mass concentrated at a point at a distance r with the force $\frac{m}{r^2}$. The equipotential surfaces are spherical surfaces, and the lines of force are straight lines proceeding from the material particle. Let us consider two equipotential surfaces, N_1 and N_2 (Fig. 20), and imagine both to be covered uniformly with a very thin layer of matter, in fact such that the unit mass shall cover the unit surface. If r_1 and r_2 represent their radii, then the equipotential N_1 contains the mass $4\pi r^2_1$, and the equipotential surface N_2 contains the mass $4\pi r^2_2$. The force with which the former

mass will be attracted is equal to $\frac{m}{r^2_1} \cdot 4\pi r^2_1$, or $4\pi m$; the force with which the latter will be attracted is equal to $\frac{m}{r^2_2} \cdot 4\pi r^2_2$ or $4\pi m$. Both forces are thus equal to one another. The force which acts on N_2 is everywhere at right angles to N_2, and in the space between N_1 and N_2 is directed inwards; the force which acts on N_1 is everywhere at right angles to N_1; it is moreover directed outwards from the space between N_1 and N_2. We will now consider any closed surface whatever, and cover it with a very thin layer of matter in such a way that the unit mass covers the unit surface, and we will consider that force which tends to attract the masses lying on the separate portions of the surface in the space bounded by the surface, in a direction at right angles to the divisions of the surface under consideration, *i.e.* in the sense of normals drawn inwards, to be positive; on the other hand, that force which tends to do the same in an opposite direction we will assume to be negative, and will designate the sum of all these forces as the tension under which the space bounded by the surface stands. With the help of this idea, we can shortly express the result obtained above; namely, the tension under which the space between N_1 and N_2 stands is equal to zero.

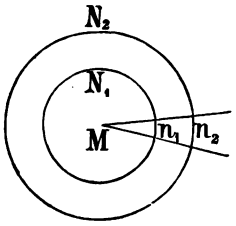

Fig. 20.

If we now imagine a cone to be constructed from the material particle M, which cuts out from the surfaces N_1 and N_2 the parts n_1 and n_2, the masses on these sections will be n_1 and n_2. The tension under which the space enclosed by n_1 and n_2 and the conical surface stands is

determined solely by the forces acting on n_1 and n_2, because the tangent planes contain the lines of force of the conical surface, and therefore no force can act at right angles to the conical surface. The force on n_1 is $\frac{m}{r^2_1} \cdot n_1$, and the force on n_2 is $\frac{m}{r^2} \cdot n_2$.

Now, because both these forces act in a perpendicular direction to the divisions of surface concerned, and are as regards the space under consideration in opposite senses, the tension under which the space mentioned stands will be equal to

$$m\left(\frac{n_1}{r^2_1} - \frac{n_2}{r^2_2}\right).$$

Now the divisions n_1 and n_2 have the same ratio to one another as the spherical surfaces of which they form a part, or as $4\pi r^2_1 : 4\pi r^2_2$, or as $r_1 : r_2$; it therefore follows that

$$\frac{n_1}{r^2_1} = \frac{n_2}{r^2_2},$$

or

$$m\left(\frac{n_1}{r^2_1} - \frac{n_2}{r^2_2}\right) = 0,$$

i.e. the tension is equal to zero. What form, however, do these relations assume, if we consider any kind of closed surface?

Let us denote the surface by S (Fig. 21), and let us again consider this to be covered with a very thin layer of matter, so that the unit mass covers the unit surface. We further construct from M a very small cone, which cuts out from the surface the parts *ab* and *cd*.

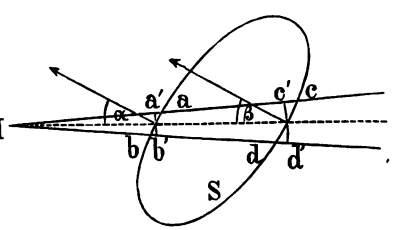

Fig. 21.

The masses of the layer on these parts possess respectively the magnitude ab and cd. If we denote by r_1 and r_2 the distances of the centres of gravity of both these portions of the surface from M, then the forces with which these will be attracted by M are equal to $\frac{ab \cdot m}{r^2_1}$ and $\frac{cd \cdot m}{r^2_2}$. In order to determine the tension under which the space enclosed by the surface S and the cone stands, we must ascertain the components of the forces acting on the portions of surface ab and cd at right angles to them. These are evidently $\frac{ab \cdot m}{r^2_1} \cos \alpha$ and $\frac{cd \cdot m}{r^2_2} \cos \beta$. Now, $ab \cos \alpha$ is nothing else than $a'b'$, or that part of the equipotential surface passing through the centre of gravity of ab, and which is cut out by the cone, and for the same reason $cd \cdot \cos \beta$ is equal to $c'd'$. Bearing this in mind, we obtain therefore for the components of the forces the expressions $\frac{m \cdot a'b'}{r^2_1}$ and $\frac{m \cdot c'd'}{r^2_2}$. Now, since none of the forces act at right angles to the surface forming the envelope of the cone, then for the same reason as before the tension under which the space enclosed by the surface S and the cone stands is equal to

$$m\left(\frac{c'd'}{r^2_2} - \frac{a'b'}{r^2_1}\right),$$

or, since the proportion $c'd' : a'b' = r^2_2 : r^2_1$ holds, is equal to zero. If from M we construct all possible conical surfaces, we can divide the space of the surface S entirely into spaces of the character of the one investigated. Each one stands under zero tension, therefore the whole of the spaces bounded by S is under zero

tension. We arrive in this manner at the following result: *If a material particle lies outside a closed surface of any form whatever, the tension under which the space bounded by the surface stands is equal to zero.*

If there lies not only one material particle, but any number outside the surface S, the tension is still equal to zero, and it is at the same time quite immaterial whether the material particles are separate or are united into a continuous mass.*

If we convert the proposition, it will run as follows: *If the tension on the space bounded by a closed surface is different from zero, the acting material particles cannot lie altogether outside the surface.*

What is now the character of the relation which exists between the magnitude of the tension and the magnitude of the mass at the enclosed material point?

Let us now consider a closed surface S (Fig. 22) in which a material particle M lies, and let us imagine this surface to be covered with a very thin layer of matter in such a manner that the unit mass falls on the unit surface. If we construct from M a cone, this will cut out from the surface an area *ab* on which the mass *ab* lies. If the centre of gravity of this portion of the surface is distant from the point M, the length r, the force with which the mass M will be attracted, is equal to $\frac{M \cdot ab}{r^2}$.

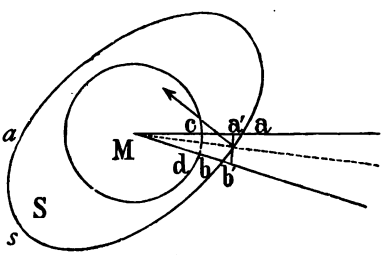

Fig. 22.

* In the mathematical theory of potential that differential equation which expresses this notion is usually designated as Laplace's theorem.

In order to obtain its share in the tension we have to determine the component perpendicular to the portion of surface *ab*. This is $\frac{M \cdot ab}{r^2} \cos \alpha$. But $ab \cdot \cos \alpha$ is nothing else than $a'b'$, *i.e.* that division of the equipotential surface passing through the centre of gravity of *ab*, and which is cut out by the cone; that component is therefore equal to $\frac{M \cdot a'b'}{r^2}$.

If we construct about the point M a sphere of radius 1, then that part of the sphere which is cut out by the cone (we will call it α) has the same ratio to $a'b'$ as 1 to r^2, or α is equal to $\frac{a'b'}{r^2}$. The component of the force considered therefore takes the form $M\alpha$. I can consider the division of the surface α as the projection of the part of surface *ab* on the spherical surface of radius 1, and may arrive therefore at the result that the share of the part of surface *ab* in the tension under which the spaces bounded by S stands, is found if the mass M of the material particle is multiplied by the projection of the part of the surface on the spherical surface of radius 1. As is evident, the share $M\alpha$ considered is positive, because the force is directed towards the interior of the surface. If we give to the cone all possible positions we can investigate the shares of all the divisions of the surface S in the tension. Because M is situated in the interior of the surface the force belonging to all parts of the surface is directed inwards; its share is therefore positive.

The sum of all the shares, *i.e.* the tension under which the space bounded by S stands, is consequently equal to the mass M multiplied by the sum of the projections of the divisions of the surface on the

spherical surface of radius 1; or, because this last sum amounts to the whole spherical surface, the sum of the projections consequently amounts to 4π and the tension is equal to $4\pi M$, *i.e.* in words: *If a closed surface of any kind whatever surrounds a material particle of the mass* M, *the space bounded by it stands under the tension* $4\pi M$.

If several material particles are situated in the interior each one produces a tension. If we sum all these tensions the total tension is found, and this is equal to 4π multiplied by the sum of all the enclosed masses. Conversely we may say, if we divide the tension under which the space bounded by the enclosed surface stands by 4π, we obtain the masses situated in the space.

These propositions hold in a similar way, whether we have to deal with isolated material particles or with a continuously distributed mass, since the latter can always be supposed to be composed of countless numbers of material particles. If we consider a closed surface *a*, within a continuously distributed mass (Fig. 23), the tension under which the space bounded by it stands, so far as it proceeds from the masses situated outside, will be equal to zero, but in so far as it proceeds from the enclosed mass it possesses the value $4m\pi$ if *m* represents this latter mass. Or, conversely, the mass *m* is equal to the tension under which the space enclosed by *a* stands divided by 4π. If *a* encloses a very small space, we can consider the quotient of *m* divided by this space as the density at the observed position, and arrive therefore at the proposition: *If we imagine a small closed surface constructed in a continuously distributed mass, then the tension under which the space*

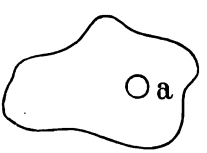

Fig. 23.

bounded by the surface stands, divided by 4π *times the contents of the space, is the density at that position.**

If the continuously distributed mass is a very thin material layer, we can conceive this case as if the mass were spread over one of the bounding surfaces, and then in this uniform distribution of mass that mass which covers the unit surface is comprehended under the term density. If the distribution of the mass is not uniform, then that mass, which would cover the unit surface if the distribution of the mass at all points of the surface were the same as at the point considered, defines the density at a point. We obtain the magnitude of the density at a point if we divide the amount of the mass lying on an adjacent very small surface by the area of this portion of surface.

We can here, as previously, discover a relation between the density and the acting forces. Let the very thin material layer be represented by the surface S (Fig. 24). Let us imagine a very small part of surface *ab* to be enclosed by a curve, and let lines of force be drawn through all the points of the boundary line, and, parallel to *ab* but very near to it, let two surfaces be constructed. The lines of force cut out from these last two portions of surface $a'b'$ and $a''b''$, which we, since they lie very near one another, can assume to be equal and also of the same magnitude as *ab*. If we now further suppose $a'b'$ and $a''b''$ and the surface composed of the lines of force to be covered uniformly with a very thin material layer, in such a way that the unit mass covers the unit surface, then we can ascertain the tension under which the space bounded by the surfaces

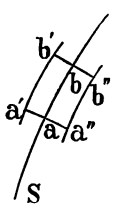

Fig. 24.

* In the mathematical theory of potential, the differential equation which expresses this idea is named Poisson's theorem.

in question stands if we determine the forces acting at right angles to the various portions of surface.

No force acts at right angles to the surface formed by the lines of force; there remain only the portions of surface $a'b'$ and $a''b''$ to consider. These portions of surface contain the masses $a'b'$ and $a''b''$. If P′ and P″ are the forces at $a'b'$ and $a''b''$ acting at right angles to ab on the unit mass, then the forces $P'.a'b'$ and $P''.a''b''$ are those that act on the masses $a'b'$ and $a''b''$ at right angles to ab. Let the direction of $P'.a'b'$ pass inwards in the space and that of $P''.a''b''$ outwards in the same. The tension under which the space stands is therefore

$$P'.a'b' - P''.a''b'',$$

or

$$(P' - P'')ab.$$

According to the foregoing, however, the tension is equal to 4π multiplied by the mass contained within the space, or, if we denote the density at ab by d, it is equal to $4\pi . ab . d$.

We therefore obtain the equation

$$(P' - P'')ab = 4\pi . ab . d,$$

or

$$d = \frac{1}{4\pi}(P' - P''),$$

i.e. *the density of any point in a very thin material layer is equal to* $\frac{1}{4\pi}$ *multiplied by the difference of those forces which act on the unit mass at the point considered, on both sides of the surface, at right angles to the surface, the two forces being both reckoned in the same sense.*

SECTION 20.—**Potential of two Material Particles.**

Let two material particles M_1 and M_2 be of the same mass, viz. the mass M, and let their distance apart be equal to a.

If the unit mass, which we represent to ourselves as concentrated at a point, comes from infinity to a point situated at a finite distance, the point being at the distance r_1 from M_1 and r_2 from M_2, then the work done is given by $\frac{M_1}{r_1}$ and $\frac{M_2}{r_2}$ respectively, and the total work of attraction by the magnitude

$$\frac{M}{r_1}+\frac{M}{r_2},$$

or

$$M\left(\frac{1}{r_1}+\frac{1}{r_2}\right),$$

i.e. the total potential due to the two material particles

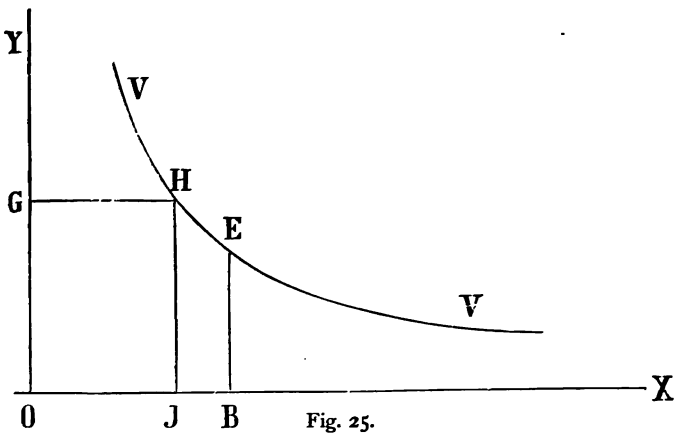

Fig. 25.

is equal to the sum of the potentials due to the single material particles considered separately.

A surface at all points of which the same potential prevails is called an equipotential surface. The equipotential surfaces of a single material particle are concentric spherical surfaces. In the case, however, where we have to deal with two or more material particles,

the equipotential surfaces are in reality differently shaped. The following is a simple method of constructing the equipotential surfaces of two material points. We construct the curve $V=\frac{M}{r}$ by marking off on the straight line OX (Fig. 25) the distance $OB=r$, at the end of this distance erecting a straight line BE at right angles to it, and making this equal to that value of the potential V which follows from the equation $V=\frac{M}{r}$ for the given value of r. If we join the terminal points of these perpendiculars we obtain the curve VV. In order to obtain the value of V we write the equation in the form $V.\ r=M$, or $V:\sqrt{M}=\sqrt{M}:r$, and represent this graphically in the following manner. We erect on

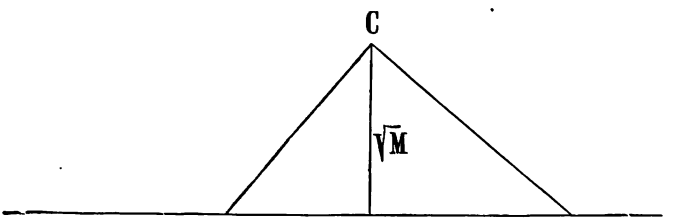

Fig. 26.

a straight line AA (Fig. 26) at the point O′ a perpendicular and make this equal to $\sqrt{M}$, we mark off from O′ on the one side the distance $O'B=r$, we join B and C and draw through the point C a straight line at right angles to BC, *i.e.* CD. O′D is then the value of V corresponding to r. If we erect a line equal to this distance in Fig. 25 at the point B at right angles to OX, we obtain the point E. If we give all possible values to the space O′B and always adopt the same

construction as in Fig. 26 we obtain in succession in the space O'D the corresponding V, and if we erect these in Fig. 25 as perpendiculars, and join their terminal points, we get the curve VV.

In the foregoing case we have two exactly equal material particles; the same curve VV therefore holds for both. An equipotential surface, as has been stated, is characterised by this, viz. that on it the sum $V_1 + V_2$ possesses one and the same value. If we have two points fixed in space, A and B, and if we seek for the points C (Fig. 27) for which AC + CB, *i.e.* the sum of the distances from A and B, has always the same value, we obtain, as a geometrical locus of these points, a ellipsoid of revolution. If we cut this by a plane which passes through the points A and B, we obtain an ellipse as the figure of the section. If we now assume that the constant value which the sum $V_1 + V_2$ will have along an equipotential surface is equal to the constant value of the distances AC + CB, then we can imagine V_1 represented by the length AC and V_2 by the length BC. If we cut off on the straight line OY from the point O a distance OG = AC, and through G draw a parallel to OX, this will cut the curve VV at a point H, and if we then let fall from H the perpendicular HJ on the straight line OX, then HJ represents the potential V_1 and OJ the corresponding r_1. If we do the same with the distance BC, we in the same

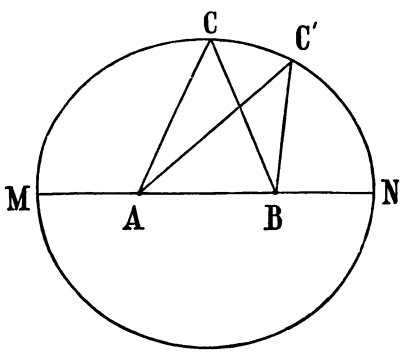

Fig. 27.

way find r_2 corresponding to V_2. If we now describe round the point M_1 a circle with radius r_1, and round the point M_2 a circle with radius r_2, we obtain at the points of intersection the potential $V_1 + V_2$ of the given value. Let us now cause the point C to gradually move round the ellipse, and when, for example, it comes to C', taking the distance AC' as V, let us find by the method given above the corresponding value of r_1, and similarly the value of r_2 corresponding to BC' considered as V_2, and let us describe again two circles with the radii r_1 and r_2 round the points M_1 and M_2 respectively; we thus again obtain two points of intersection at which the sum $V_1 + V_2$ possesses the given value, and so on.

In this way we by and by obtain a row of points of the corresponding equipotential surface, which when joined together give that curve in which the equipotential surface is cut by the plane (plane of the sketch) passing through M_1 and M_2 (Fig. 28). All the surfaces are symmetrical about the axis M_1M_2; whatever position the planes of section may have, the curve of section is always the same. If we imagine therefore the curve rotated round M_1M_2 as axis, it generates the equipotential surface.

The equipotential surfaces are surfaces of constant potential, or otherwise expressed surfaces of the same work-value. If we move the point at which we imagine the unit mass to be concentrated along such an equipotential surface, the work-value will not be changed by this—it remains the same; or, otherwise expressed, the increase of the work of the attracting forces is equal to zero. The work is, however, as we have seen, equal to the product of the displacement and that component of the acting force which corresponds with

the direction of displacement. If this product is equal to zero, but the displacement different from zero, then must that component have zero value, or the working force, when it is different from zero, must be at right angles to the direction of displacement. Inasmuch as the displacement takes place on the equipotential surface, the force must be at right angles to the equipotential surface.

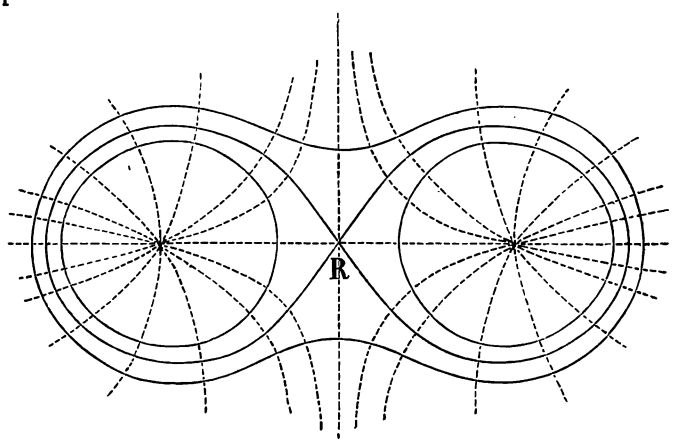

Fig. 28.

If the material particle is free to follow the force of attraction, it moves everywhere at right angles to the equipotential surfaces, or cuts through these everywhere in a perpendicular direction. The course which the material point thereby describes is called a line of force. If we give to the material particle gradually all possible initial positions, and on each occasion leave it to itself, we shall obtain an infinite number of such lines of force. All run out at right angles to the equipotential surfaces.

At the point R an equipotential surface cuts itself; what does this mean? If at the point R in our figure

we apply tangents to the equipotential curve and imagine the whole figure turned about the axis M_1M_2, the two tangents generate a conical surface with the vertex at R. There is no straight line which stands at the same time at right angles to all the generating straight lines; there is therefore no force at the point R. If a material particle is brought to the point R, it does not move, it remains in a state of rest. The point R is a *point of rest, a point of equilibrium.* This is easily intelligible: the point R lies in the middle of the line joining the material particles M_1 and M_2, therefore the forces of attraction acting at that place must be equal in magnitude but opposite in direction, and consequently neutralise each other.

All equipotential surfaces on which the potential possesses a value different from zero are closed surfaces. If such a surface were not closed it must extend to infinity. At an infinite distance, however, as we have seen above, the potential of every material particle has a zero value. The surface must consequently at the portion situated at an infinite distance have zero potential, and hence must have this value of potential at all the remaining points, since it is a surface of equal potential, which is contrary to the supposition.

Since the equipotential surfaces are closed surfaces, they successively envelop each other. The nearer an equipotential surface lies to a material particle, the greater is the corresponding value of the work of the force of attraction, or the potential. Each equipotential surface divides space into two divisions; the one contains higher, the other lower values of potential. Two different equipotential surfaces can never cut one another, because there can be no point at which the potential has two different values.

We will now consider the equipotential surfaces of two material particles when the masses of the particles are unequal and stand to one another in the ratio 1 : 2. We can make use of the curve $V = \frac{M}{r}$ (Fig. 25) in the construction for the first material particle without further

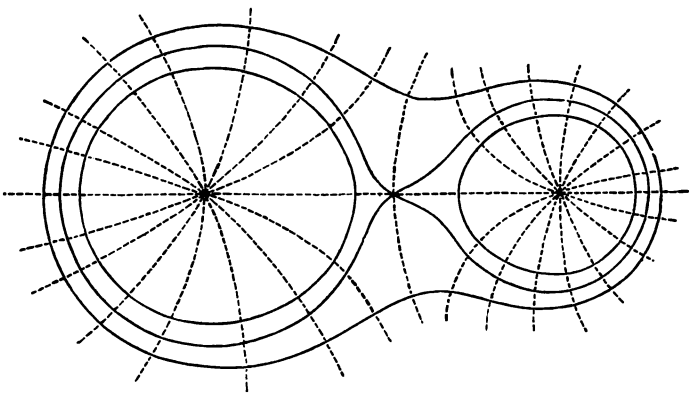

Fig. 29.

trouble, and form a second curve from this by extending the ordinates r for this in the ratio 1 : 2. We find again on the straight line joining the material particles a point P at which an equipotential surface cut itself. This is as before a point of equilibrium (Fig. 29).

CHAPTER II

ELECTRICAL POTENTIAL (STATICAL ELECTRICITY)

SECTION 21.—**The Fundamental Phenomena of Statical Electricity.**

IF a glass rod is rubbed it acquires the property of attracting small light bodies; it is said to be electrified or electrical. If the elder pith ball I (Fig. 30), which is suspended from a fixed point A, is touched with such a rod, this ball likewise acquires the property of attracting small light bodies; it likewise becomes electric.

If the elder pith ball II, which is suspended from the fixed point B, is touched with the glass rod, this also becomes electrical.

Between I and II now, however, there occurs a reciprocal action; the two balls repel one other. If the glass rod is again brought near both balls, then both will be repelled; if, however, a rubbed rod of resin (stick of sealing-wax) is brought near them, both will be attracted. If at first the balls are touched with a rubbed rod of resin instead of with a rubbed glass rod, they will again both become electric and again mutually repel one another. If, however, the one ball is touched with a rubbed glass rod and the other with a rubbed rod of resin, they mutually attract one another. If both balls

are charged with the same "kind of electricity" or "of like name," they mutually repel one another; if, however, they are charged with a different "kind of electricity" or "of unlike name," they mutually attract one another. If in the latter case we bring both balls into mutual contact, they will afterwards either not attract or repel one another at all, or they will only feebly repel one another. The electrical states of both balls have therefore either neutralised one another, or they have changed into a very weak state of the same name; or, in other words, both balls, by their contact, acquire the same state, such indeed, that the former states either mutually neutralise one another, or some of what was in excess remains. We therefore obtain the following proposition.

Fig. 30.

"*Vitreous electricity*" *and* "*resinous electricity*," *as regards the action of their force, both on* "*vitreous electricity*" *as well as on* "*resinous electricity*," *exhibit an entirely opposite character. The* "*vitreous electricity*" *is changed by the addition of* "*resinous electricity*," *just as if a negative number were added to a positive. For this reason the former is designated as positive and the latter as negative electricity.*

If, when both balls are electrified with the same kind of electricity, I hold the point A fixed, and move the point B towards A, I observe that the repulsion becomes greater and greater; if I withdraw B from A, the repulsion will become less and less. The force with which the two balls act on one another depends on their distance apart; it is greater if this is smaller, and conversely.

SECTION 22.—**Coulomb's Law.**

The law according to which the force of electrical repulsion or attraction respectively changes with distance was first ascertained experimentally by Coulomb (1785). The apparatus which he employed for this purpose bears the name of the "torsion balance," and depends on the following principle.

At the point A (Fig. 31) let a very fine silver wire be fastened, and at the other end B of the same let a very slender small rod of shellac or a glass thread be applied. If we turn this small rod from its position of equilibrium round the wire AB as axis, the wire AB will experience a twist, a torsion, which resists the turning of the small rod. This resistance is of the same nature as the action of a couple whose moment increases with the torsion of the wire. By a couple is understood two equal and parallel forces, but tending in opposite directions (Fig. 31), and whose tendency is to rotate the straight line CD. The measure of this tendency to rotation is given by the product of one of the forces and the perpendicular CE, let fall from C on to the prolongation of the other force, *i.e.* into the distance between the two forces. This product is called the moment of the couple. The moment of the couple, which has its origin in the torsion of the thread, increases proportionally with this, *i.e.* it becomes 2, 3, 4 times as great if the torsion becomes 2, 3, 4 times as great.

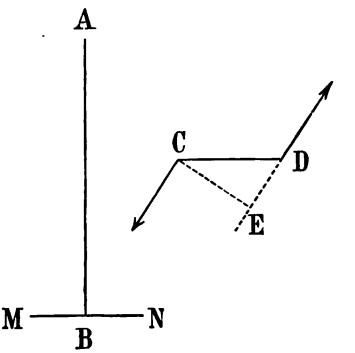

Fig. 31.

If we place at M a small gilt elder pith ball, and fix opposite this by means of an insulating shellac rod another gilt ball of the same size, so that the straight line joining the centres of the two balls is at right angles to the direction MN, then, as soon as both balls have been charged with electricity of the same name, the fixed ball will repel the one at M. The thin rod MN will be made to revolve, and by this means the wire will be twisted, until the moment of the repelling force which acts on the arm of the lever MB is held in equilibrium by the moment of the couple. In Fig. 32 we find the apparatus sketched. *m* is a fixed ball ("stationary ball"), *n* the other ball, which is balanced by a small plate of mica placed vertically at the other end of the small rod. The small rod revolves within a scale on the circumference of the cylinder. The upper end of the silver wire is fastened to a knob which is placed in a brass disc. The latter possesses on its circumference a graduated scale on which a pointer, fixed to the knob, indicates.

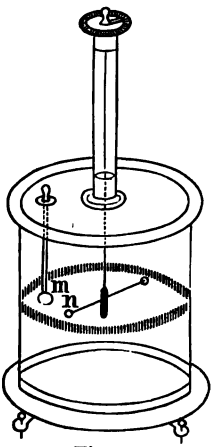

Fig. 32.

The experiment is carried out in the following manner: The knob is placed in such a position that the ball *n* touches the ball *m* without the slightest torsion of the silver wire. The ball *m* is then taken out, electrified, and again brought inside. Through contact it gives up a portion of its electricity to the ball *n*; in consequence of this a repulsion at once occurs. Let the angle of deflection of the rod be *a*. The knob to which the silver wire is attached is now turned, so that the distance between the two balls may become less.

If δ is the angle through which the knob was turned, and if the small rod comes to a position which makes with the original position of equilibrium the angle a', then the wire has experienced a torsion equal to $a' + \delta$. If the turning of the knob is still carried on in the same direction, until the total revolution of the knob amounts to δ', the torsion of the wire—if the small rod is deflected from the original position of equilibrium through the angle a''—will thereby amount to $\delta' + a''$. Coulomb found the following numbers amongst others:

$$a = 36^\circ \qquad a = 36^\circ$$
$$a' = 18^\circ \qquad \delta + a' = 144^\circ$$
$$a'' = 8.5^\circ \qquad \delta' + a'' = 575.5^\circ$$

The distances between the balls are nearly proportional to the deflections of the small rod, hence as

$$36 : 18 : 8.5,$$

or nearly as

$$4 : 2 : 1.$$

The repelling forces are proportional to the torsion of the wire, hence as

$$36 : 144 : 575.5,$$

or nearly as

$$1 : 4 : 16,$$

or

$$1^2 : 2^2 : 4^2.$$

The forces of repulsion therefore vary inversely as the squares of the distances. Coulomb likewise proved that this law holds also for attraction. Does this law not remind us of Newton's law of gravitation? May we not here again express the result by representing as the seat of electric force masses which exert force on each other proportional to the product of the masses, and inversely proportional to the square of the distance? The dependence stated, of the force on the distance, does not appear from Coulomb's experiments with the torsion balance mentioned above, with complete

clearness, since this is bound up with many sources of error; it is, however, as we shall see, given with complete precision owing to the experimentally ascertained fact that no trace of electrical force is found in the interior of a conductor. Although this latter experiment has been carried out only when the interior of the conductor is filled with air, we may with certainty expect the same result when the interior is filled with any other insulator. *If we represent masses as carriers of electric forces, these, by whatever insulator they may be separated from one another, must always act on one another according to the inverse square of the distance.*

SECTION 23.—The Electrostatic Unit of Electric Quantity.

Just as, in the case of heavy bodies, we deduce masses from weights, *i.e.* from gravitating forces, so we deduce the magnitude of electrical masses from the laws of force named above. We measure forces when we compare them with a force taken as the unit. The centimetre is agreed upon as the unit of length, as unit of mass the mass of a cubic centimetre of pure distilled water at 4° centigrade, and as unit of time the second. The unit mass named is designated as "a gram," and the weight of this for the purpose of distinction as "a gram-weight." Now, since force is given by the product of mass and acceleration, we must regard *as the unit of force that force which imparts to the unit of mass the unit of acceleration*, i.e. *the increase of velocity per second of one centimetre per second.* The acceleration of gravity is, for the sea level at the equator, very nearly 978 centimetres. The force with which the unit of mass (one gram) is attracted there is 978 units of force.

The force with which, at the same place, a cubic millimetre of pure distilled water at 4° C.—this is called a mass of "one milligram"—is attracted, is a thousand times as small, therefore equal to 0.978 unit of force, or conversely the unit of force is equal to the weight of 1.02248 milligram, therefore roughly equal to a milligram-weight. Work is equal to the product of force and the projection of the displacement on the direction of the force. The unit of work is then accomplished if the unit of force produces a displacement of one centimetre in the direction of the force. If the unit of mass falls vertically downwards at the equator through a distance of one centimetre, then 978 units of work will be done. If a milligram falls freely through a distance of one centimetre, then the work of gravity is nearly equal to the unit of work.

If we now imagine two very small balls, each charged with the same amount of electricity of like name and at a distance apart of one centimetre, and if we represent to ourselves that the force with which the balls are repelled is the unit force, we obtain, if we denote the quantities of electricity by m, the equation

$$1 = \frac{m \cdot m}{1^2},$$

or

$$m = 1.$$

By the electrostatic unit of electricity we therefore understand that quantity of electricity which repels an equal quantity of electricity of like name at the unit distance with the unit force.

In order to form a notion of the mutual action of two electrostatic units, we will consider a simple pendulum (Fig. 33), which consists of an insulating thread of 978 centimetres in length, and a very small ball of the unit mass. The force with which this ball is attracted

at the equator is equal to 978 units of force. The force, however, which tends to drive the ball back, by reason of its angular displacement α, to its position of equilibrium, is equal to this force multiplied by sin α. If the angle α is very small we can put the ratio $\frac{BC}{AB}$ for sin α, or $\frac{BC}{978}$, and we obtain consequently for the latter force the value $978' \times \frac{BC}{978'}$, or BC units of force. If BC amounts to one centimetre, the force is just equal to the unit of force. While the pendulum remains in the position AC, let us imagine a small conducting ball of the same magnitude as C, brought to the point B, held fast there, and then both balls charged with the electrostatic unit of electricity. The force with which the ball B repels the ball C is now equal and opposite to the force with which the earth urges the ball C towards B. The two forces neutralise each other, and the ball C remains in equilibrium in the position indicated.

Fig. 33.

The phenomena of repulsion described above can occur just as well by the employment of negative as by the employment of positive electricity. The units of positive and negative electricity are of the same magnitude but of opposite sign. This is expressed by saying: the positive unit of electricity $\times -1$ is equal to the negative unit, hence the positive unit of electricity can be considered as the unit by means of which all electricities, whether positive or negative, are measured. If we measure quantities of electricity, we express the positive by positive numbers, the negative by negative numbers.

Let there be two very small conducting spheres,

one of which is charged with the quantity of electricity e, the other, at the distance r, with the quantity of electricity e_1, then the force of the mutual action is equal

$$\frac{ee_1}{r^2}.$$

If e and e_1 are both positive or both negative numbers, then the expression for the force is positive, and we have repulsion. If e is positive and e_1 negative, or conversely, then the expression for the force is negative, and we then have attraction. The attraction is therefore to be looked upon as a negative repulsion. Quite naturally, for repulsion acts in the sense of increasing, attraction in the sense of diminishing r. Moreover, if a force which acts in the sense of increasing r is to be reckoned as positive, then must a force which acts in the sense of diminishing r, and therefore acts in an opposite sense, be reckoned as negative.

SECTION 24.—**Potential of an Electrified Point.**

The law which applies to the mutual action of two electrified points—always understanding thereby very small spheres which are charged with electricity—is the same as Newton's law of the force of gravitation; for electric forces, therefore, relations and laws result entirely analogous to those with which we have become acquainted in regard to gravitation.

We have seen that the work of a force is positive if the force forms an acute angle with the direction of displacement, and therefore if the projection of the force on the direction of the displacement acts in the sense of the displacement; on the other hand negative, if the force forms an obtuse angle with the direction of the

displacement, or the projection of the force on the direction of displacement has a sense opposite to the latter. We find an example in illustration of this in free fall and in vertical upward throw. In the former case gravity acts always in the same direction as the displacement—the work is therefore positive; in the latter, on the other hand, gravity is directly opposed to the motion, and seeks to hinder it—the work therefore is negative.

Two electrified points of the same name repel one another. If the points freely obey the force, the force acts always in the sense of the displacement; the work is then positive. If we, however, keep the one point fixed and bring the other near to it, then the force of repulsion acts against the motion and the work is

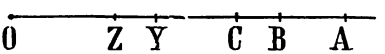

Fig. 34.

negative. Two electrified points of unlike name attract one another. If the points freely obey the force, the force always acts in the sense of the displacement and the work is positive. If we, on the other hand, hold the one point fast and withdraw from it the other, the force always acts contrary to the motion and the work is negative.

Now let us consider an electrified point O (Fig. 34), which we imagine to be charged with the unit of positive electricity. The lines of force are evidently straight lines; one of these is the straight line OA. If we suppose the unit of electricity brought to the point Z and exposed to the force of repulsion, it will successively arrive at the points C, B, A. . . . If we say, we bring the unit of positive electricity to the point Z,

that is the same as saying that we bring a very small ball which is charged with the unit of positive electricity to the same point. The work which the force of repulsion performs during the motion from Z to A is evidently the same which, in the case of section 17, Fig. 19, the force of attraction performs by the motion from A to Z, for in both cases the force acts in the sense of the displacement and changes from point to point according to the same law. It has therefore, if we denote the distances of the points A and Z from O by a and z, the magnitude

$$\frac{1}{z} - \frac{1}{a}$$

If the unit of positive electricity moves beyond the point A and arrives at infinity, then the work of the repelling force is equal to

$$\frac{1}{z}.$$

If we now imagine the unit of positive electricity brought back from infinity along the straight line OA, by the application of any force whatever, to its former position Z, then during the whole motion the electric force of repulsion acts in a sense opposite to the motion. Its work is now negative, but of the same absolute magnitude as before, therefore

$$-\frac{1}{z}.$$

If the point O were charged with the positive quantity of electricity e, then the work observed would be e-times as great, or equal to

$$-\frac{e}{z}.$$

If the point O were charged with the quantity of

electricity -1, then the electric force would act in the sense of the motion during the passage of the unit of positive electricity from infinity to the point Z; its work would be, because attraction is really negative repulsion, equal and opposite to that work which we formerly obtained while investigating the unit of positive electricity at the point O, *i.e.*

$$\frac{1}{z},$$

or

$$-\frac{-1}{z}.$$

If the quantity of electricity at the point O possessed the magnitude $-e$, then would the work be e-times as great, or

$$\frac{e}{z},$$

or

$$-\frac{-e}{z}.$$

That all these quantities of work are independent of the direction follows immediately from the observations aforesaid in regard to gravitation.

The work which the forces proceeding from an electrified point perform when the unit of positive electricity is brought from infinity to a given point, is named the potential due to the electrified point at the given point. The magnitude of this potential is obtained if the quantity of electricity of the electrified point is divided by the distance of the given point and the negative sign prefixed to the quotient.

To every point in space there belongs a definite potential value. The difference of the potential value of two points is named the potential difference of these points. The unit of potential difference prevails between two points when the electrical forces perform the

unit of work when the unit of positive electricity is moved from the one point to the other.

All points which have the same value of potential form an equipotential surface. The equipotential surfaces of an electrified point are concentric spherical surfaces. The different equipotential surfaces are distinguished by different values of potential. If the electrified point is charged with positive electricity, the potential is everywhere negative ; at an infinite distance, however, it is equal to zero. The nearer the equipotential surfaces lie to the electrified point, the smaller is the value of potential. If the electrified point is charged with negative electricity, the potential is then everywhere positive ; at an infinite distance, however, it is equal to zero. The equipotential surfaces possess the greater value of potential the nearer they lie to the electrified point. *If the unit of positive electricity can freely follow the lines of force proceeding from an electrified point, it then moves, whether the electrified point be positive or negative, from places of lower potential to places of higher potential.*

The lines of force of an electrified point are, as already stated, straight lines, which radiate from the point. By the electrical or electromotive force at a point, we will for the future understand that force which acts on a very small sphere, which, when charged with the unit of positive electricity, is brought to that point, without the distribution of the electrical masses having, in consequence, been disturbed.

Let two points which lie near one another have the values of potential A and B. Then $A - B$ is the work of electric force during the motion of the unit of positive electricity from the second point to the first. If we divide the difference $A - B$ by the distance

between the points, we obtain the component of electric force acting in the straight line joining the points or in the direction of displacement; in other words, if the unit of positive electricity moves from one point to another point very near the first, then the component electric force acting in the direction of motion is given by the quotient of the increase of potential by the magnitude of the displacement. If the potential diminishes, then the increase of potential is negative, and the quotient is negative, *i.e.* the component acts in a direction opposite to the displacement with a force which is given by the absolute value of the quotient.

From what has been said it follows that an electric force exists between two points, or, in other words, that by the action of the electric masses the unit of positive electricity can be transferred from one point to another only when there exists between these a difference of potential.

SECTION 25.—Electrical Separation (Influence).

We have defined the electric force at a point as that force which acts on a very small sphere, which, when charged with the unit of positive electricity, is brought to that point. If no further force acts on the sphere, and if it can freely follow the tendency of the electric force, it will move in the direction of the force. If we again bring to the same point a very small sphere, but charged with the unit of negative electricity, and if the same conditions exist as before, the sphere now moves in the opposite direction. Inasmuch as the electric force causes here a motion of bodies, it is called ponderomotive.

Let us now imagine a so-called conductor, *i.e.* a body

on which the electricity can follow, with complete freedom, the tendency of the electric force brought into the space outside the electrical point O (Fig. 35).

Let the conductor be charged with no electricity, but be in the so-called neutral condition. We can represent to ourselves such a condition by supposing that all points of the conductor are charged with electricity, viz. with positive and negative electricity in equal degree, so that both these completely neutralise each other as regards their outward action.

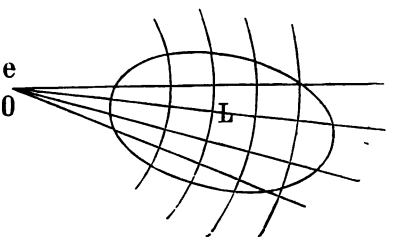

Fig. 35.

The conductor, as we see in Fig. 35, is penetrated by a number of equipotential surfaces, to each of which a different value of potential belongs. There appear electrical forces, which tend to separate the two electricities united at each point of the conductor and to move them in the direction of the lines of force, the positive in the one, and the negative in the opposite sense. If we draw from the point O all possible lines of force, and consider the points at which these pass through the conductor, the two electricities at every one of such points will follow freely the tendency of the electric force, the positive in the one, the negative in the opposite sense, since the conductor offers no resistance to their motion. The electric forces effect therefore, at the various points of the conductor, a separation of the two electricities, and in consequence are called electromotive forces.

If the point O is charged with positive electricity, then negative electricity collects on the conductor on

the side turned to O, and on the opposite side negative. If the point O is charged with negative electricity, the distribution of electricity on the conductor is reversed. The entire result is designated by the name "influence," and that electricity which is found on the part of the conductor turned towards O is named "influence electricity of the first kind." Since both these influence electricities originally mutually neutralised one another, they must be exactly equal in magnitude.

We have designated those bodies as conductors on which a quantity of electricity can freely follow the electric force; those are called non-conductors or insulators on which this is not the case. On these also the single points must be looked upon as charged with both electricities in equal amount, but the electrical force cannot, as on a conductor, produce such a separation that the positive electricity moves from the point in the one sense, the negative in the other.

In the theory of electrical phenomena only complete conductors and complete non-conductors are considered. In reality such a separation of bodies does not exist; a gradual transition from bodies which insulate completely to those bodies which conduct completely exists. That distinction is made in theory merely because by its help the laws of electrical phenomena can be much more simply expressed. On similar grounds in mechanics, solid bodies are considered to be completely rigid, liquids as completely incompressible; for the same reason in optics, the circumstance that all transparent bodies absorb light more or less is often neglected, and so on.

With the help of what has been stated above about electrical separation, we can in a simple manner make visible the direction of the electrical force at every point of space. Let us imagine that we had a very thin, very

light, elongated conductor—for example, a thin piece of wire, suspended at its centre by an insulating silk thread, and the centre brought to that point at which we wish to determine the direction of the electrical force. Owing to the effect of the electromotive force, both ends of the conductor will be oppositely charged and attracted in opposite directions, so that the conductor assumes a position in which its axis coincides with the direction of the electrical force.

We can determine in a similar manner those points of space which belong to the same equipotential surface. Let us imagine two very small conducting, but uncharged, spheres to be suspended by insulating silk threads. Let us bring the first to the point A, the second to the point B (*i.e.* we make the centres of the spheres coincide with these points). Let the points A and B possess a different potential. If we join the two spheres by a conducting wire held by an insulating handle, and then withdraw the wire, both spheres appear charged, and indeed to an equal extent and opposite in sign. That is to say, since the centres of both spheres are at points which have different values of potential, an electromotive force has made its appearance which moves the positive and negative electricity in different senses, and in this manner charges that sphere on which the potential has the greater value positively ; on the other hand it charges the other negatively. If we hold the sphere situated at A fast in its place, and move the other sphere round the point A, we shall find points such that if we join the two spheres by the wire and take it away, neither the one nor the other sphere appears charged. It is by this proved that no electrical force appears between the two spheres so placed ; therefore no difference of potential exists between their centres ;

these are consequently on the same equipotential surface. It is possible in this way to find an unlimited number of points which are on the same equipotential surface, and by means of them determine the surface itself.

SECTION 26.—Equipotential Surfaces due to two Electrified Points.

Hitherto we have supposed that the electrical or electromotive forces proceeded from one electrified point. If the charge at this is e, then the potential at the point at the distance r is $-\frac{e}{r}$. The potential depends only on the charge e and the distance r. If we have to deal with two electrified points at which the charges e_1 and e_2 are found, then the work of the electrical forces proceeding from these electricities when the unit of positive electricity is transferred from infinity to a given point, will be expressed in a manner entirely analogous to that in section 20, by

$$-\frac{e_1}{r_1}-\frac{e_2}{r_2},$$

where r_1 and r_2 denote the distances of the given point from the electrified point.

If the quantities of electricity e_1 and e_2 are of equal magnitude and of the same kind (thus both positive or both negative), the potential can be written in the form

$$-e\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$

The equipotential surfaces are in this case, at their intersection with a plane going through both electrified points, of the same form as those equipotential surfaces which we have constructed in section 20 (Fig. 28).

If we put $e_1 = 1$ and $e_2 = 2$, we get for the course of the equipotential surfaces the same representation as in Fig. 29. If the two quantities of electricity are of the same magnitude, but of unlike kind, $e_2 = -e_1$, the potential has then the form

$$-e\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If we now construct the equipotential surfaces at their intersection with a plane which passes through both electrified points, they assume the form shown in Fig. 36.

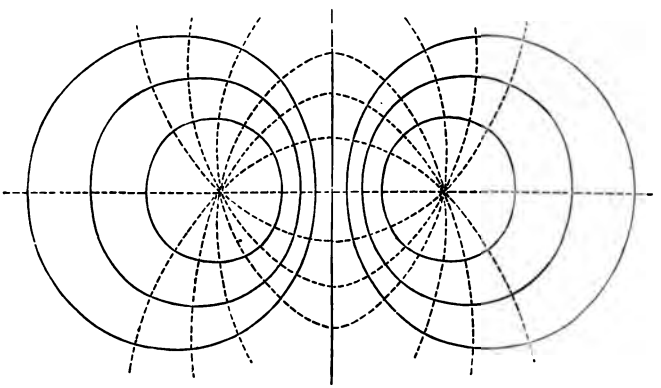

Fig. 36.

In order to construct this figure, we again start from the curve $y = \frac{e}{r}$. We do not, however, take an ellipse as a further subsidiary figure, but a hyperbola. If we write

$$\frac{e_1}{r_1} = y_1, \quad \text{and} \quad \frac{e_2}{r_2} = y_2,$$

then the potential is equal to $y_2 - y_1$.

If y_1 and y_2 denote the distances from two fixed points, all the points for which this difference has the

same value will lie on a hyperbola. We construct therefore a hyperbola, we draw radii vectores to any point of the same, suppose these to be y_1 and y_2, we seek with the help of the curve $y = \frac{e}{r}$ the distances r_1 and r_2 for them, and describe with these distances as radii circles round the electrified points. At the points of intersection of the circles we obtain points of the same equipotential surface. By repeating this process we can obtain any number of points of the same equipotential surface. If we join these, we obtain the curve in which the equipotential surface is cut by a plane (plane of the drawing) passing through the electrified points. The equipotential surface itself is formed by turning the curve of the plane section round the straight line joining the electrified points as axis.

One of the equipotential surfaces is a plane at right angles to the connecting straight line mentioned, and bisects this. Since it stretches to infinity, where the potential has zero value, the potential must have zero value at all points of the plane. And, in fact, the equation of this plane is given by $r_2 - r_1 = 0$, or $\frac{1}{r_2} - \frac{1}{r_1} = 0$. On all other equipotential surfaces, consequently, a value of potential different from zero prevails; no part therefore can stretch to infinity—they must all be closed surfaces.

The equipotential surfaces cannot cut themselves in one or in several points, because the potential in no point of space can have two or more different values at the same time.

The plane of zero potential divides space into two parts; in the one are negative, in the other positive values of potential. It further divides the equipotential surfaces into two groups. Since the equipotential

surfaces are closed surfaces and do not cut each other, the equipotential surfaces of each group must then successively envelop each other. Each one of these divides space into two parts; in the one the potential has a higher, in the other a lower value.

If we move the unit of positive electricity on an equipotential surface, the electrical forces do no work, because all points of the equipotential surface have the same potential. The electrical forces are therefore everywhere in a direction at right angles to the equipotential surfaces, or the system of lines of force everywhere passes through the system of equipotential surfaces at right angles. At the points of the plane to which zero potential belongs, the lines of force run parallel to the straight line, connecting electrified points.

If we have three electrified points which are charged with electrical quantities, e_1, e_2, e_3, then the potential, at a point which is distant r_1, r_2, r_3 from the electrified points, consists of three quantities of work corresponding to the electrical forces which proceed from the three electrified points. Each portion of work is represented by the form $-\frac{e}{r}$, therefore the potential at the point has the expression

$$-\left(\frac{e_1}{r_1}+\frac{e_2}{r_2}+\frac{e_3}{r_3}\right).$$

And if we have a great number of electrified points, we obtain the potential at a given point if we divide the charge of each one of these electrified points by its distance from the given point, add all the quotients so formed, and prefix to the sum the negative sign. Naturally here the positive charges must be expressed by positive, and the negative by negative numbers. If we have an electrical mass of continuous extension (such as, for example, covers the surface of a conductor with-

out break), we then find the potential of this mass at a given point if we imagine the mass to be divided into an unlimited number of infinitely small portions of mass, and divide each one of them by its distance from the given point, add all the quotients so formed, and to the sum prefix the negative sign. Then again the positive quantities of electricity must of course be expressed by positive, the negative by negative numbers.

SECTION 27.—Distribution of Electricity on a Conductor.

In section 19 we obtained the following proposition: "If all acting heavy masses lie outside a closed surface, the space bounded by it stands under zero tension." "If the mass M is inside a closed surface, the enclosed space stands under the tension $4\pi M$." The law which connects force with distance is, as regards gravity and electricity, the same; we can therefore transfer both the propositions enunciated at once to the domain of electricity if we suppose the closed surfaces uniformly covered with positive electricity, so that the unit quantity falls on the unit surface; as regards the sign, however, which is employed in calculating the tensions corresponding to the various portions of the surface, the opposite is adopted. We will now stipulate that the forces which urge the masses lying on the various portions of the surface in the sense of the normal drawn outward shall be reckoned as positive; those forces, on the other hand, which urge them in a contrary direction shall be reckoned as negative. There is besides only one thing still to consider: If the surface encloses positive and negative quantities of electricity of equal amount, the sum of these masses equals zero; the tension

therefore under which the space stands is likewise zero. We can, in conformity with this, express the proposition already stated in the following manner: The space of a closed surface, which we suppose covered uniformly with positive electricity in such a way that the unit surface is covered with the unit mass, experiences, by the action of the given electrical masses, zero tension, if these masses lie either entirely outside the surface, or if the surface encloses positive and negative masses in equal amount. The space on the other hand experiences a tension different from zero if the surface encloses either merely positive or merely negative masses, or positive and negative masses in unequal amount.

Now suppose we had again at a point O (Fig. 37) a positive amount e of electricity, and that a conductor were brought into the neighbourhood of the same. At the various points of this, as we have already seen in section 25 (Fig. 35), forces of separation arise which charge the conductor with influence-electricity. Since the charge of the point O is positive, the influence-electricity of the first kind is negative, that of the second kind positive.

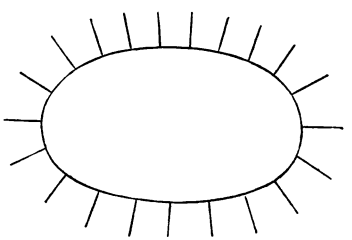

Fig. 37.

The separation of the electricities does not go on without limit. As the separated quantities of electricity have come forth from the points in which they formerly were, they act directly against the separating force of the electrified point O at the former point; additional negative electricity separated at any point, and seeking

to move itself in the direction towards O, will be repelled by the negative electricity already separated and attracted by the positive electricity already separated. And exactly the same occurs as regards additional positive electricity separated at any point. The separated electricities act in the opposite sense to the electromotive force of the point O, and so much the more strongly the greater the quantities of electricity that are separated. The electromotive or separating forces therefore diminish at the separate points of the conductor in proportion as the separation proceeds, and must finally altogether disappear. The separation, and therewith the motion of the electricity on the conductor, ceases, and the electrical masses are then found on the conductor in a condition of equilibrium.

We have seen that electric force acts between two points only when these points exhibit a difference of potential. In the condition of equilibrium mentioned there is no electric force existing; the points of the conductor have therefore also no difference of potential, or all points of the conductor have the same potential value. This potential value is called the potential of the conductor.

The points of the conductor behave just as the points of an equipotential surface. The points of the surface of the conductor also possess the same value of potential, and indeed the same as the points in the interior. The surface of the conductor is therefore an equipotential surface. In the interior, the electric forces are equal to zero, but outside the conductor they are different from zero. Since the surface of the conductor is now an equipotential surface, the electric force in the immediate neighbourhood of the same is in a direction at right angles to it, *i.e.* the lines of force of the

external space all meet the surface in a perpendicular direction.

Now the question occurs, Where have the separated quantities of electricity betaken themselves? If we imagine a closed surface S, of any form, constructed in the interior of the conductor and covered uniformly with positive electricity, so that the unit of mass covers the unit of surface, the space bounded by this will be subject to zero tension, because the electric force at all points of the conductor has zero value. We conclude from this that in the interior of the surface there are either no electrical masses or positive and negative electrical masses in equal amount. Let us imagine further that the surface S contracts until the space bounded by it is infinitely small. The tension under which the space stands will always have zero value, because in reality the electrical force at all points of the conductor has zero value, and therefore also when the space has become infinitely small. In consequence of what precedes, we must also now conclude that in an infinitely small space either no electrical masses are present or positive and negative masses in equal quantity. This, however, comes to the same thing; for if I imagine to myself positive and negative masses in equal quantity concentrated at a point, their external effects are completely neutralised. The point appears uncharged. The unelectrified condition is, as we have seen, supposed to be due to the presence of positive and negative electricities united together in equal amount, because only under this assumption can the phenomena of influence be explained. We can assume the infinitely small space to which S has contracted, and which has proved to be unelectrified, to be anywhere in the interior of the conductor; the conductor will consequently at

all its points in the interior appear to be unelectrified; it will nowhere in the interior contain any quantity of electricity whatever. Well, then, where are the separated quantities of electricity? They are on the surface, the negative on the side turned towards the point O, the positive on the side opposite to the negative.

The fact that electricity merely exists on the surface has been experimentally established for a very long time. The first experiments in this direction are due to Coulomb (1786-1788). The following experiment of Coulomb's is generally known: A metallic sphere is supported by an insulating glass pedestal. Two half spherical shells of metal supplied with insulating glass handles can closely enclose the sphere. If the sphere is now covered with these and these latter electrified, the sphere, on the withdrawal of the hemispheres, shows no trace of electricity. Among the numerous researches which were undertaken in this direction after Coulomb, Faraday's are to be specially noticed. One of them is the following: Faraday caused a light cube to be made, consisting of a wooden frame, which he caused to be wound round with a netting of copper wire, and this again he had covered with tinfoil. Faraday entered the cube with a very delicate electroscope, and caused the cube, which was well insulated, to be strongly charged. In the interior no trace of electricity made its appearance.

Mach performed this experiment in a very simple manner: A gold-leaf electroscope (Fig. 38), with a spherical glass vessel, was placed in a larger cylinder of glass and covered with a hood of conducting material. Water is now poured into the vessel until the level is very little distant from the hood. If an electrified body now approaches, the leaves diverge. If water is now poured in until the hood is wet, the hood can be

touched with a body, however strongly electrified it may be, without the leaves showing any divergence.

By these experiments that theoretic conclusion, which is deduced on the basis of Coulomb's law, is established. But these experiments do still more. It can, for instance, be mathematically demonstrated that if the interior of a charged conductor shows no trace of electricity, Coulomb's law of the mutual action of the electrical masses follows as a necessary consequence. The methods available for determining whether a body is charged with electricity or not are of a most delicate kind; the above experiments are therefore to be considered as completely decisive, and by them Coulomb's law is completely verified.

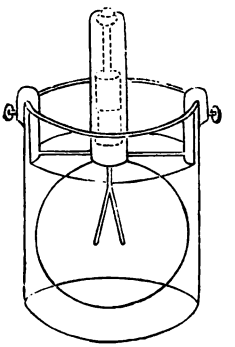

Fig. 38.

SECTION 28.—**Continuation: the Density on the Surface of a Conductor.**

In the foregoing it was shown that electricity collects only on the surface; we must, in conformity with this, consider the electrical charge of a conductor as an exceedingly thin layer. We obtain the density of this at the various points of the surface, if we make use of the relations given at the end of section 19. We there obtained the proposition: The density of a very thin layer of matter at a given point of the same is equal to $\frac{1}{4\pi}$ multiplied by the difference of the normal forces (both reckoned in the same sense) which act on the unit

of mass at that point on the two sides of the layer. If we substitute in this proposition the word "unit of positive electricity" for "unit of mass," we can immediately adopt the proposition: The density of the electrical layer at a given point is equal to $\frac{1}{4\pi}$ multiplied by the difference of the normal forces (both reckoned in the same sense) which act on the unit of positive electricity at that point on the two sides of the layer. If, however, the electrical forces at all points of a conductor are equal to zero, the component of force acting on the inner side of the surface and at right angles to it is equal to zero. It is further to be kept in view that an equipotential surface is everywhere met by lines of force at right angles to it. The electrical force is therefore at every point situated outside of the surface, and infinitely near to it, at right angles to the surface, or that component of force which, outside of the surface, acts at right angles to the surface, is the whole force. If we, finally, further keep in view what we said at the beginning of the last paragraph about the sign of the components of the tension corresponding to the various parts of a closed surface, we obtain for the electrical density on the surface of a conductor the following proposition: *The electrical density at a point of the surface of a conductor is equal to* $-\frac{1}{4\pi}$ *multiplied by the electrical force at this point. If this force is directed outwards the density is positive, if it is directed inwards the density is negative.*

The electrical density on the surface of a conductor and the potential of the conductor are two entirely different things. The former depends, as we have seen, on the force, *i.e.* on the rate of fall of the potential at

the surface in the direction of the normal. The conductor has at all points the same potential; on the other hand the density of the charge at the different points of its surface is commonly different.

SECTION 29.—Continuation: Application to a Conducting Sphere.

Let a conducting sphere be charged with a quantity of electricity Q, and let the space outside the sphere be completely free from any quantity of electricity whatever. The electricity under these circumstances spreads itself uniformly over the surface and forms a very thin homogeneous spherical shell. In section 10 and section 18 we saw that the potential of a homogeneous heavy spherical shell at all points of the interior hollow space possesses the same value, and that outside the spherical shell it has exactly the same value as if the whole mass were concentrated in the centre of the spherical shell. Now, since electrical masses act, as regards distance, according to the same law, in our case the potential outside the sphere will be the same as if the whole electrical charge were concentrated at the centre, and therefore it is of the value $-\frac{Q}{r}$, if r denotes the distance of the point under consideration from the centre of the sphere. If a is the radius of the sphere, then the potential has the value $-\frac{Q}{a}$ on the surface of the sphere and at all points in the interior. The charge considered to be concentrated at the centre, and which outside the sphere produces an effect equal to that of the actual distribution of electricity on the surface, is designated as the *electrical image* of this distribution.

The electrical force in the interior of the sphere is equal to zero; outside the sphere, however, it follows the same law as if it proceeded from the electrical image; that is, it possesses the magnitude $\frac{Q}{r^2}$. At points in the surface it amounts to $\frac{Q}{a^2}$; we obtain, therefore, for the density, by the application of the proposition given above,

$$\frac{Q}{4\pi a^2}.$$

If Q is positive, then in the first place the force, which acts on the unit of positive electricity on the surface, is directed outwards; in the second the density is positive. If Q is negative, then the force is directed inwards, and the density is negative. The density in this case is found without having recourse to the assistance of the above rule. If for instance the quantity of electricity Q is distributed uniformly on the surface, then, because the surface of the sphere amounts to $4\pi a^2$, there falls on the unit surface the quantity $\frac{Q}{4\pi a^2}$, and this is nothing else than the density.

Let us now imagine a conducting spherical shell charged with a positive quantity of electricity Q, and very far removed from all other conductors. The electricity then, under such circumstances, spreads out uniformly and produces thereby in the interior not the smallest force; the potential is the same at all points of the interior hollow space. Let us now suppose an uncharged conductor A to be in the interior of the spherical shell. Since the electrical force at all points in the interior has zero value, no separation will take place in the conductor A; it therefore remains uncharged. On

the other hand, however, the potential at all points of the interior, therefore also at the points of the conductor A, has a constant value different from zero, and indeed has the value $-\frac{Q}{a}$, if a denotes the radius of the spherical shell. From this we see that a conductor, without containing the least trace of electricity, can become raised to a potential different from zero. If we charge the spherical shell with a continuously increasing quantity of electricity, *the absolute value of the potential on the conductor* A *will continue to rise; the charge of the conductor remains, however, always equal to zero.*

SECTION 30.—**Sphere and Electric Point.**

At the point B (Fig. 39) let there be an electrical mass e concentrated, and let it act by influence on a conducting sphere whose centre is at A, and which is completely free of charge. The electrical forces produce a separation of electricity in such a manner that the influence-electricity of the first kind collects opposite the point B, the influence-electricity of the second kind collects on the farther side of the sphere. If the condition of equilibrium has occurred, then all the electrical masses are found on the surface, and the potential has at all points in the interior a constant value. We will determine this value.

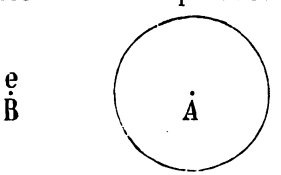

Fig. 39.

Since the sphere was originally unelectrified, and on separation positive and negative electricities originate from the various points in equal amount, there will be present on the surface equal quantities of positive and negative electricity.

The potential which these quantities and the quantity concentrated at the point B produce, will, as has already been stated, have the same value everywhere in the interior of the sphere. We consider first of all this value for the point A. The potential due to the electrical point B is at that place $-\frac{e}{AB}$, where AB is the straight line joining A and B. As regards the potential due to the influence-electricity, we obtain this, if we (section 26) imagine the masses to be broken up into an unlimited number of infinitely small portions, divide each of them by its distance from A, *i.e.* by the radius of the sphere, add the whole of the quotients so formed, and prefix the negative sign to the sum. In the case of all the terms of this sum the denominator is common; we obtain therefore the value of the sum if we add the numerators and divide this sum by the common denominator. The sum of the numerators is now nothing else than the sum of the quantities of electricity on the surface of the sphere. These consist, however, of equal portions of positive and negative electricity; the sum is therefore zero, and consequently also the potential of the influence-electricities at the point A is zero. We obtain therefore for the potential at the point A the value

$$-\frac{e}{AB},$$

or $-\frac{e}{AB}$ is the *potential of the sphere.*

If the condition of equilibrium has set in on the sphere, all the influence-electricity is on the surface—in the interior there is no trace of it; the state of equilibrium will therefore not be changed if I suppose the interior portion of the sphere taken out, and imagine

the complete sphere made into a hollow sphere or a spherical shell.

Now the potential must still have everywhere the same value in the interior of the hollow sphere. If I move the point B to an infinite distance, then both the influence-electricities unite and the sphere becomes unelectrified. If I move the point B from infinity again to its former position, the same conditions will occur as before. If we thus have from the first a given hollow sphere, and if an external electrified point acts on this, producing influence-electricities, the potential, after the establishment of the condition of equilibrium, possesses the same value not only at all points of the hollow sphere, but also at all points of the interior hollow space.

SECTION 31.—**Continuation: Application to the Earth.**

If the sphere under consideration is our earth, and the electrified point B is situated at the place where we are ourselves,—that is, at a distance from the earth's surface, which is infinitely small in comparison with the radius of the earth,—we may without hesitation make the distance AB equal to the earth's radius R, and obtain for the potential of the earth the expression

$$-\frac{e}{R}.$$

The numbers by which we denote the quantities e with which we experiment are in general very small in comparison with that number by which the earth's radius (636,620,000) is denoted; the fraction $\frac{e}{R}$ therefore, and consequently the potential of the earth, is considered as

exceedingly small. It can, moreover, be easily shown that the potential of the earth is not simply nearly equal to zero, but is altogether equal to zero. The potential to which the earth is raised by a positive electrical quantity e, which is on its surface—and only this position comes within our consideration in experiments—is equal $-\frac{e}{R}$. Now it is, however, known that everywhere when electricity is developed, both electricities appear in equal quantity. To every positive quantity e there corresponds a precisely equal negative quantity $-e$, which raises the earth to the potential $+\frac{e}{R}$.

Even when the two electricities in the course of the experiment separate from one another, yet the distance of both from the centre of the earth is still to be looked upon as equal; they always produce on the earth values of potential of equal amount but of opposite sign; *the combined potential is therefore zero.*

We may thus, in all our experiments, consider the earth as a place at which zero potential prevails, and can in conformity with this *define* the potential at a given point *as the work which the electric forces perform when the unit of positive electricity arrives at that point from the earth.*

SECTION 32.—Continuation: the Quantity of Influence-Electricity of the first kind, in the case of connection with the Earth.

If we connect the sphere considered in section 30, when electrical equilibrium has taken place on it, with the earth (Fig. 40) by means of a conducting wire, then,

because the earth possesses zero potential, while the sphere possesses a potential different from zero, in consequence of this potential difference electrical forces arise in the wire, which urge the positive electricity from places of lower potential to places of higher potential, but the negative electricity, on the other hand, in the opposite sense. If the quantity of electricity at the point B is positive, then the potential value of the sphere is negative, consequently less than the potential of the earth; the positive electricity will flow from the sphere to the earth, as the arrow shows—the negative, on the other hand, from the earth to the sphere, and this will continue as long as a difference of potential exists. Whilst by this the negative influence-electricity of the first kind continuously increases, it diminishes the effect of the electrical force of the point B at the points of the sphere, of the wire, and of the earth, and finally neutralises it entirely. The effect of the positive influence-electricity, which spreads itself on the earth, does not enter into consideration. If the electrical force has now become equal to zero everywhere at the various points of the sphere, of the wire, and of the earth, then there prevails everywhere the same potential, viz. zero potential, because the potential of the earth has always zero value.

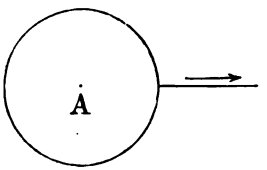

Fig. 40.

With the help of these results we can very easily calculate the quantity of negative influence-electricity, The positive influence-electricity has, as already stated. spread itself on the surface of the earth, and in consequence of this produces at the points of the sphere a

potential which can be considered as equal to zero. The electrical point B and the negative influence-electricity which is on the surface of the sphere alone determine the potential at the various points of the sphere. Let us consider the potential at A. At this point the potential due to the electrified point B has the value $-\frac{e}{AB}$, and the potential due to the influence-electricity the value $\frac{E}{a}$, if we denote by $-E$ the quantity of the influence-electricity and by a the radius of the sphere; the potential therefore which results from both has the value

$$\frac{E}{a}-\frac{e}{AB}.$$

Now, since this has zero value, we obtain the equation

$$\frac{E}{a}=\frac{e}{AB}, \quad \text{or} \quad \frac{E}{e}=\frac{a}{AB},$$

i.e. *the absolute value of the influence-electricity of the first kind is to the absolute value of the influencing electricity of the point* B *as the radius of the sphere is to the distance of the point* B *from the centre of the sphere.*

If we write the last equation in the form

$$-E=-\frac{e}{AB}\cdot a,$$

we can enunciate it as follows: *The influence-electricity of the first kind is equal to the product of the radius and the potential due to the electrical point at the centre of the sphere.*

If we have several electrified points given outside the sphere and the sphere is connected with the earth, the influence-electricity, which makes its appearance on the sphere in consequence of each one of those points, is found in the same manner, viz. as the product of the

radius and the potential due to the electrified point at the centre of the sphere. The influence-electricity, which is produced on the sphere by all the electrified points present, is consequently equal to the product of the radius, and the potential due to all the electrified points at the centre of the sphere. As a matter of course this holds good completely when the electrical masses situated outside the sphere have a continuous distribution.

The influence-electricity, the radius of the sphere being the same, is the greater the greater the potential is at the centre of the sphere, and in the case of the same value of this potential is greater the greater the radius.

In the result just arrived at concerning the influence-electricity we have at the same time found a means for determining the various points of an equipotential surface. If, for instance, we move a very small conducting sphere which is always connected with the earth in the neighbourhood of electrical masses, so that the influence-electricity is always of the same amount, then the potential of the electrical masses at the centre of the sphere has always the same value, or the centre of the sphere moves on an equipotential surface.

SECTION 33.—Sphere and Concentric Spherical Shell.

Let A (Fig. 41) be a conducting sphere enclosed by a concentric conducting spherical shell, which we will suppose to be very thin. Let the radius of the sphere be a, and that of the spherical shell b. Let us imagine further that the sphere A is charged with a certain quantity of positive electricity e, and that

the spherical shell is connected with the earth by a wire W. Let this wire, for the sake of simplicity, be rectilinear and so attached that its prolongation goes through the centre of the sphere. Since everything is symmetrical about the centre the electricity will spread itself on A uniformly and produce a potential which, at an external point, that is, distant r from the centre, possesses the value $-\frac{e}{r}$. All concentric spherical surfaces are equipotential surfaces for this potential, consequently also the spherical shell K. The potential which the sphere engenders on this has the value $-\frac{e}{b}$. Because the spherical shell is in conducting communication with the earth, which is at zero potential, there is a difference of potential between the two, and consequently a current takes place along the wire.

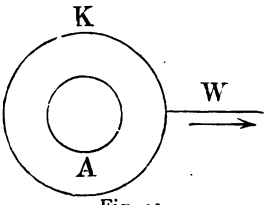

Fig. 41.

The positive electricity moves from places of lower potential to places of higher potential, hence from the spherical shell, which possesses a negative value of potential, to the earth; the negative electricity, on the other hand, in an opposite sense, and continues to do so until the spherical shell is brought to the level zero. On the spherical shell, then, only negative electricity is found, the influence-electricity of the first kind, which, like the electricity on the sphere, possesses a uniform distribution. If $-E$ is the quantity of this, and r the distance of an external point from the centre, then at that point the potential of the influence-electricity has the value $\frac{E}{r}$. On the spherical shell it

has the value $\frac{E}{b}$, and the same value also at all points of the interior.

The potential of the sphere A on the spherical shell is, as we have seen, $-\frac{e}{b}$; therefore the total potential there is

$$-\frac{e}{b}+\frac{E}{b}.$$

Now, since the shell is connected with the earth, it must be at potential zero, whence it follows that

$$E=e,$$

i.e. *the influence-electricity of the first kind is equal to the quantity of the influencing electricity* e.

At a point outside the spherical shell, distant r from the centre, the potential of the charge e is equal to $-\frac{e}{r}$, and that of the influence-electricity is equal to $\frac{E}{r}$; therefore the total potential is equal

$$\frac{E}{r}-\frac{e}{r} \text{ or } 0.$$

At all points outside the spherical shell the potential as well as the electrical force has zero value.

In the interior of the spherical shell the potential of the charge e, at a point outside the sphere at the distance ρ from the centre, has the value $-\frac{e}{\rho}$; on the other hand the potential of the influence-electricity has the value $\frac{E}{b}$ or $\frac{e}{b}$; hence the total potential is

$$-e\left(\frac{1}{\rho}-\frac{1}{b}\right).$$

On the surface and at all interior points of the

sphere A the potential takes the value $-e\left(\frac{1}{a}-\frac{1}{b}\right)$. This is negative because a is less than b. If we interrupt the connection with the earth, the potential of the spherical shell remains at the level zero. If we now, however, charge the spherical shell with a quantity of negative electricity $-E'$, then this will spread out uniformly and produce in the interior the potential $\frac{E'}{b}$. At the points on the sphere the potential will by this means be brought to

$$\frac{E'}{b}-e\left(\frac{1}{a}-\frac{1}{b}\right).$$

If we let E' go on increasing from zero, this potential continually increases, will become zero and finally positive. We can alter the potential of the sphere A as we choose without the charge experiencing the slightest alteration. We see from this that *the sign of the potential of a conductor need not on the least depend upon the sign of the charge.*

SECTION 34.—The Spherical Shell encloses an electrified point in any position—Electrical Images.

We now consider a conducting spherical shell which is in connection with the earth, and an electrified point with the charge e enclosed by it, which point is outside the centre.

The elecrified point e acts influencingly (inductively) on the spherical shell. If e is positive, then the influence-electricity is negative. This collects on the spherical shell whilst that of the second kind flows off. We will denote the quantity of the influence-electricity

by $-E$ and ascertain its amount. For this end we must, however, make somewhat further investigations.

Let us imagine at two points, P, P′ (Fig. 42), two quantities of electricity of opposite sign to be concentrated, viz. at P the quantity e_1, at P′ the quantity $-e_2$. The potential of these quantities at a point which is r_1 distant from P and r_2 from P′ possesses the value

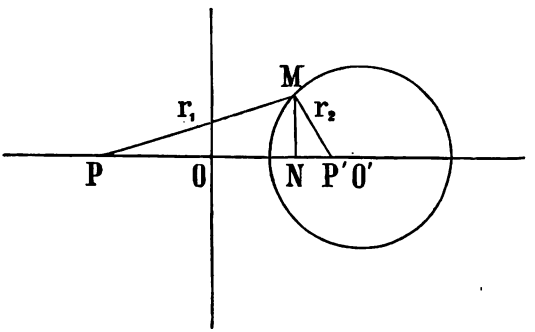

Fig. 42.

$-\frac{e_1}{r_1}+\frac{e_2}{r_2}$, and consequently the equation of an equipotential surface will be expressed by

$$-\frac{e_1}{r_1}+\frac{e_2}{r_2}=\text{const.}$$

That equipotential surface which corresponds to the potential zero is given by

$$-\frac{e_1}{r_1}+\frac{e_2}{r_2}=0,$$

or

$$\frac{e_1}{r_1}=\frac{e_2}{r_2},$$

or

$$\frac{r_1}{r_2}=\frac{e_1}{e_2}=m,$$

where m represents the constant ratio of the quantities of electricity e_1, e_2.

In order to ascertain this equipotential surface, we bisect the distance PP′ at the point O, and let fall from the point M the perpendicular MN on PP′. If we put $OP = OP' = c$, $MN = y$, and $ON = x$, there follows from the triangle PMN

$$r_1^2 = (c+x)^2 + y^2,$$

and from the triangle P′MN

$$r_2^2 = (c-x)^2 + y^2,$$

or by division

$$\frac{r_1^2}{r_2^2} = \frac{(c+x)^2 + y^2}{(c-x)^2 + y^2},$$

or

$$m^2 = \frac{(c+x)^2 + y^2}{(c-x)^2 + y^2},$$

or $m^2y^2 + m^2c^2 + m^2x^2 - 2m^2cx = c^2 + x^2 + 2cx + y^2,$

or

$$(m^2-1)y^2 + (m^2-1)x^2 - 2c(m^2+1)x + (m^2-1)c^2 = 0;$$

or, if we divide the whole equation by $m^2 - 1$,

$$y^2 + x^2 - 2c\frac{m^2+1}{m^2-1} \cdot x + c^2 = 0.$$

This equation represents a circle whose centre is on the straight line PP′ at the distance

$$\frac{m^2+1}{m^2-1} \cdot c$$

from the point O, and which is described with a radius equal to

$$\sqrt{\left(\frac{m^2+1}{m^2-1}\right)^2 \cdot c^2 - c^2} = \frac{2m}{m^2-1} \cdot c.$$

Let the centre of this circle be O′. Because

$$PO' = PO + OO' = c + \frac{m^2+1}{m^2-1} \cdot c = \frac{2m^2}{m^2-1} \cdot c,$$

and

$$P'O' = OO' - OP' = \frac{m^2+1}{m^2-1}c - c = \frac{2}{m^2-1} \cdot c,$$

then $$\frac{PO'}{P'O'} = m^2.$$

If we name the radius of the circle a, then is

$$\frac{2mc}{m^2 - 1} = a,$$

consequently

$$PO' = ma, \quad P'O' = \frac{a}{m},$$

and therefore

$$PO' \cdot P'O' = a^2.$$

From the equation $PO' = ma$ it follows that

$$\frac{e_1}{e_2} = \frac{PO'}{a}.$$

Let us imagine (Fig. 43) a conducting very thin spherical shell of radius a connected with the earth by a wire W, and a quantity of positive electricity e concentrated at the point P. In consequence of the influence which e causes, influence-electricity of the first kind—this is here negative—collects on the surface of the spherical shell opposite the point P, whilst the positive influence-electricity goes off to the earth. On account of its connection with the earth the spherical shell has the potential zero. This is equal to the sum of the potential due to the quantity e and of the potential due to the influence-electricity. A quantity of negative electricity concentrated at the point P', situated on the straight line PO in such a position that $P'O = \frac{a^2}{PO}$, and

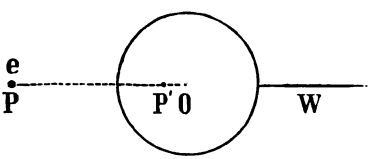

Fig. 43.

having the magnitude $-\frac{a}{PO}.e$, would also produce precisely the same potential which the negative influence-electricity produces at the various points of the spherical shell, for such an electrified point in conjunction with the electrified point P would produce zero potential on the spherical shell. And if we imagine conversely that the quantity of electricity $+\frac{a}{PO}.e$ is brought to the point P', this point produces exactly the same potential on the spherical shell as the electrified point P. For this reason the electrified point P' is designated the *electrical image of the point* P.

Let us now remove the quantity of electricity e and cause it to move to infinity; the charge of the spherical shell disappears, the spherical shell becomes unelectrified. When this has happened we bring to the point P' the quantity of electricity $+\frac{a}{PO}.e$. This influences the sphere; the negative influence-electricity collects on the spherical shell, the positive influence-electricity goes off to the earth, and the potential of the spherical shell is again equal to zero. This potential is given by the sum of the potentials due to the electrified point P' and the potential due to the influence-electricity. Now since the former is equal to the potential due to the point P, so also is the latter equal to the potential due to the former influence-electricity, and this holds for all points of the spherical shell. It follows therefore from this that in the present case influence-electricity has collected on the spherical shell in the same manner and to the same amount as in the previous case.

The amount of this, however, according to section

32, was $-a.\frac{e}{OP'}$, therefore the present influence-electricity also possesses this magnitude, or *it is equal and opposite to the charge of the point* P'. We therefore arrive at the following proposition: *If there is in the interior of a conducting spherical shell—the position is optional—an electrified point, and if the spherical shell is connected with the earth, then the influence-electricity of the first kind which collects on the spherical shell is equal and opposite to the quantity of electricity at the electrified point.*

If we have in the interior of the spherical shell an electrically charged body, this charge can be considered as composed of infinitely numerous electrified points; and because what has been stated above holds good of every one of these points, the influence-electricity of the first kind which collects on the spherical shell will be equal and opposite to the charge of the body.

This proposition can be illustrated experimentally in the following manner.*

Let us imagine a zone cut out from a spherical shell and this made into a lid (Fig. 44). Let an insulating silk thread pass through an aperture in the middle of this lid, and let a conducting sphere be suspended at the end of it. Let the sphere be electrically charged, for example, positively. If now the lid is lowered on to the spherical shell by means of the silk thread, so as to close the shell, the gold leaves g separate. If a rubbed stick of resin is made to approach the gold leaves, they separate still further; they are therefore charged positively. The sphere in fact influences the

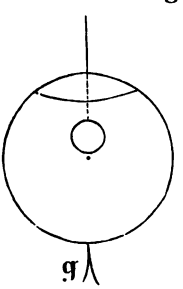

Fig. 44.

* Maxwell, *Elementary Treatise on Electricity*, Art 17.

spherical shell so that the negative electricity makes its appearance on the inner, the positive, on the other hand, on the outer surface of the spherical shell. If the spherical shell is connected by means of a conducting wire with the earth (water or gas pipe of the house), the positive influence-electricity escapes, the leaves collapse, the potential of the spherical shell acquires zero value. This value does not change if we remove the wire. When this is done, we can by means of the silk thread move the sphere up and down in any way we please in the interior of the spherical shell without the gold leaves becoming in the least separated. This is a proof that at any position whatever of the sphere the negative influence-electricity has the same magnitude. For if any position of the sphere demanded a greater amount of negative influence-electricity, a new separation must ensue and the new positive influence-electricity, which cannot now escape, must force the leaves asunder. If there were, conversely, in consequence of a certain position of the sphere, a smaller amount of negative electricity bound, the surplus would become free, and because it cannot escape would force the leaves asunder.

If we remove the sphere from the spherical shell without touching its wall, the leaves again separate—a proof that electricity was collected and bound on the spherical shell. By means of a rubbed stick of resin we can easily convince ourselves that the electricity is negative. If we again introduce the sphere into the spherical shell, the leaves collapse and remain together, even if the spherical shell is brought into contact with the sphere. Both sphere and spherical shell then show themselves to be unelectrified, from which it follows that the negative influence-electricity was the same in amount as the primary electricity.

Supplementary.—The density of the influence-electricity will in general be different at the different points of the spherical shell. Only in the case where the point P′ coincides with the centre O is the density at all points the same, and indeed on account of the complete symmetry in regard to the point P′. If e is the charge of the point P′, then in this case the density of the influence-electricity is equal to

$$-\frac{e}{4\pi a^2}.$$

SECTION 35.—**Theory of the Leyden Jar.**

The Leyden jar consists usually of a glass cylinder which is covered on both sides with tinfoil. In order, however, to discover more easily the laws in operation in the cases of the Leyden jar, we will make use of a simple conception. We will, namely, assume that the Leyden jar consists of two concentric conducting spherical shells. The laws which follow in this case hold very nearly for the actual case.

We have already, in section 33, explained the laws of two concentric spherical shells, the outer one of which is connected with the earth. We there found for the potential at an external point and at the points of the exterior spherical shell the value zero, for the potential at a point between the two spherical shells the value $-\left(\frac{1}{\rho}-\frac{1}{b}\right)e$, and for the potential on the interior spherical shell and at all points inside it the value $-\left(\frac{1}{a}-\frac{1}{b}\right)e$ or $-\frac{b-a}{ab}.e.$ Here e denotes the charge of the inner spherical shell, a and b the radii of the spherical shells, and ρ the distance of the point in question from the

centre. As we see, the potential on the inner spherical shell is smaller the smaller the intervening space between the two spherical shells.

If conversely we have determined the potential on the inner spherical shell whilst the outer is connected with the earth, and has therefore zero potential, then the quantity of electricity on the inner spherical shell is

$$e = -\frac{b}{b-a} \cdot a\mathrm{P},$$

and the quantity of electricity on the outer spherical shell is equal to $-e$,

or

$$\frac{b}{b-a} \cdot a\mathrm{P}.$$

For a given value of potential the quantity of electricity is the smaller the greater the intervening space $b-a$.

The relations shape themselves somewhat differently if it is not the outer but the inner spherical shell that is connected with the earth, and thus acquires zero potential. We may imagine the connection made by a conducting wire coming from the earth so as to go through an aperture of the outer spherical shell to the inner, without thereby touching the former. Let there be on the outer spherical shell the quantity of electricity e. This spreads uniformly and produces at all the interior points the potential $-\frac{e}{b}$.

On the inner spherical shell influence-electricity of the first kind collects whose amount we will denote by e'. Since this likewise spreads uniformly, its potential on the inner spherical shell is equal

$$-\frac{e'}{a}.$$

The total potential there, or $-\frac{e}{b}-\frac{e'}{a}$, must now have zero

value; we therefore obtain the relation

$$e' = -\frac{a}{b} \cdot e.$$

The influence-electricity is now no longer the same in amount as the influencing, but less, and indeed is less in proportion as the ratio of the radii is less. We obtain in this case for the potential at an external point, which is distant r from the centre, the value

$$-\frac{e}{r} - \frac{e'}{r}, \quad \text{or} \quad -\frac{b-a}{b} \cdot \frac{e}{r},$$

at points of the outer spherical shell the value

$$-\frac{b-a}{b} \cdot \frac{e}{b},$$

and at a point between the two shells the value

$$-\frac{\rho-a}{\rho} \cdot \frac{e}{b},$$

ρ denoting, as above, the distance of the point from the centre.

If conversely we have determined the potential P of the outer spherical shell, whilst we have brought the inner by connecting it to the earth to zero potential, we find the quantity of electricity of the outer spherical shell to be equal to

$$e = -\frac{b}{b-a} \cdot b\mathrm{P},$$

and that of the inner equal to

$$e' = -\frac{a}{b-a} \cdot b\mathrm{P}.$$

Both quantities of electricity are therefore greater in proportion as $b-a$ is smaller. If we charge the inner spherical shell with the quantity of electricity e, whilst the outer is connected with the earth, the potential difference between the two spherical shells is equal to

$$\frac{b-a}{b} \cdot \frac{e}{a}.$$

If we charge the outer spherical shell with the same quantity of electricity e, whilst the inner is connected with the earth, the potential difference equals

$$\frac{b-a}{b} \cdot \frac{e}{b}.$$

The ratio of the two potential differences is therefore as $b : a$.

By help of the foregoing expressions we can now in both cases calculate the amount of the energy or of the work-value of the charge, *i.e.* of the work which the electrical forces perform, if we bring the electric charges little by little from infinite relative distances apart to their present position. This energy consists (section 11 to section 15) of the potential of the inner spherical shell on itself, of the potential of the inner spherical shell on the outer spherical shell, and the potential of the outer spherical shell on itself. If we conceive the Leyden jar as a system, we obtain that energy or the potential of that system on itself if we multiply every portion of the quantities of electricity collected by the value of the potential prevailing there, add these products, and divide by 2. The values of potential on the spherical shells are constant; we have therefore to multiply the potential on each spherical shell by the charge found on it and take half the product. Now since zero potential occurs on one of the spherical shells, the energy is equal to half the product of the potential on the other spherical shell into the charge found on it, and is therefore in the first case equal to

$$-\tfrac{1}{2} \frac{b-a}{b} \cdot \frac{e^2}{a},$$

and in the second case equal to

$$-\tfrac{1}{2}\frac{b-a}{b}\cdot\frac{e^2}{b}.$$

If the quantity of electricity is not given, but the potential, then the energy in the first case is

$$-\tfrac{1}{2}\frac{b}{b-a}\cdot aP^2,$$

in the second case

$$-\tfrac{1}{2}\frac{b}{b-a}\cdot bP^2.$$

The energy is proportional to the square of the charge, or to the square of the potential respectively. If in both the cases considered the charge is the same, then the ratio of the energies is as $b : a$.

If on the other hand the potential is the same, then the ratio of the energies is as $a : b$.

SECTION 36.—Continuation: Connection of Jars Abreast.

We join together several jars exactly equal—let their number be n—so that the inner coats (spherical

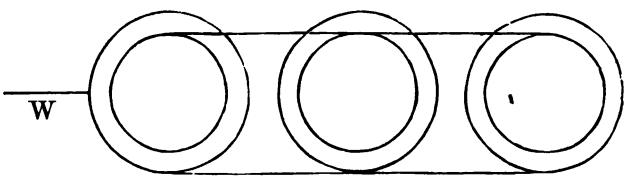

Fig. 45.

shells) are connected together and the outer coats (spherical shells) are also connected together (Fig. 45). Let the outer spherical shells be in connection with the earth by means of the wire W. Such a connection

is named a "connection abreast," or, shortly, a "battery." Let the inner spherical shells be charged. The electricity will be uniformly distributed on account of the complete equality of the jars. Let the charge of each of the spherical shells be e. If the quantities of electricity on the connecting wires be disregarded, then it may be said that the jars exert no electrical force on each other, because the potential of each at external points is equal to zero. The potential has on the inner spherical shell of each jar the value

$$-\frac{b-a}{b}\cdot\frac{e}{a};$$

on the outer spherical shells, on the other hand, the value is zero; the potential-difference between the two is consequently equal to

$$\frac{b-a}{b}\cdot\frac{e}{a}.$$

The total charge of the inner spherical shells is ne. If we have not measured the quantity of electricity but the potential, there results, if we denote this by P, for the charge the expression

$$-n\cdot\frac{b}{b-a}\cdot aP.$$

In order to calculate the energy we must keep in mind that both the inner and the outer spherical shells each form one conductor; it is therefore equal

$$-\frac{n}{2}\frac{b-a}{b}\cdot\frac{e^2}{a},$$

or

$$-\frac{n}{2}\frac{b}{b-a}\cdot aP^2.$$

The energy is proportional to the number of jars and to the square of the charge e; *or to the number of jars, and to the square of the potential.*

If we put $b = a + d$,

then $ab = a^2 + ad$.

If d is very small we can neglect in the last expression the term ad as compared to a^2. If further we denote the surface of the inner spherical shell by S ($S = 4.\pi a^2$), then the energy is given by

$$-2n\pi\frac{d}{S}e^2$$

and

$$-\frac{n}{8\pi}\frac{S}{d}P^2$$

respectively.

The energy is with a given charge, directly proportional to the distance d *of the spherical shells and inversely proportional to their surface; in the case of a given potential, on the other hand, it is directly proportional to the surface and inversely proportional to the distance* d.

The quantity d corresponds to the thickness of the glass of the common Leyden jar.

SECTION 37.—Continuation: The Franklin Battery, or the Connection of Jars in Series.

We now connect the jars in such a manner that the outer coat (spherical shell) of the first jar is united to the inner of the second jar, the outer coat (spherical shell) of this to the inner of the third jar, and so on, and finally by a wire W connect the outer coat (spherical shell) of the last jar to the earth (Fig. 46). Such a connection of the jars is named *connection in series*, or *a Franklin or cascade battery.* Let the inner spherical shell of the first jar be charged with a certain quantity of electricity. This acts by influence (inductively) on the outer spherical shell; the influence-electricity of

the first kind collects on the outer spherical shell, that of the second kind passes over to the inner spherical shell of the second jar. There it acts by influence (inductively) on the outer spherical shell; the influence-electricity of the first kind again collects on the outer spherical shell, that of the second kind passes over to the inner spherical of the third jar, and so on. At the last jar the electricity on the inner spherical shell will again act by influence (inductively) on the outer spherical shell; the influence-electricity of the first kind collects on the outer spherical shell, and that of the second kind goes away to the earth.

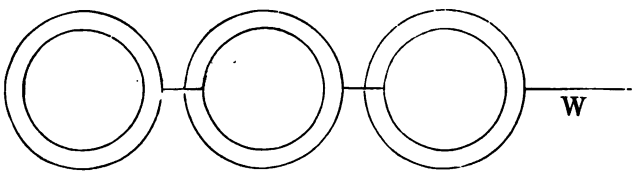

Fig. 46.

The potential of the last outer spherical shell is zero. The inner spherical shell of each jar has the same potential as the outer of the next jar following, since they are connected together so as to form one conductor. Further, the charges of these connected spherical shells, being produced by influence, are equal and opposite.

In order to be able to determine the charges and values of potential more closely in this battery, we must in the first place ascertain the relations in the case of a single jar. Let the charges of both spherical shells be uniformly distributed; let that of the inner have the magnitude e_1, that of the outer the magnitude e_2. Further, let the potential on the inner spherical shell be P_1, on the outer P_2. The charge e_1 produces on the

inner spherical shell the potential $-\frac{e_1}{a}$, on the outer the potential $-\frac{e_1}{b}$. The charge e_2 on the other hand raises both the inner and outer spherical shells to the potential $-\frac{e_2}{b}$. We obtain in conformity with this the relations

$$P_1 = -\frac{e_1}{a} - \frac{e_2}{b},$$

$$P_2 = -\frac{e_1}{b} - \frac{e_2}{b}.$$

If the values e_2 and P_2 are given we obtain from these equations the values of e_1 and P_1, viz.

$$e_1 = -bP_2 - e_2 \qquad (1).$$

$$P_1 = \frac{b}{a}P_2 + \frac{b - ae_2}{ab} \qquad (2).$$

In the second equation we will put

$$b - a = d,$$

and assume d to be so small that we can indeed neglect $\left(\frac{d}{a}\right)^2$. For P_1 we obtain by this means the equation

$$P_1 = \left(1 + \frac{d}{a}\right)P_2 + \frac{d}{ab}e_2 \qquad (2a).$$

We have, as stated, n jars. Let the potential of the inner spherical shell of the nth jar be denoted by P_{n1} and the potential of the outer spherical shell by P_{n2}. We will here throughout employ the method of adding to P, denoting the potential, two figures indicating position; the first gives the place of the jar in the series, and the additional figure, 1 or 2, denotes the inner or outer spherical shells respectively. The same method in every respect is used to denote the charges. In

accordance with what has been above stated about the charges and potentials, we in the first place obtain the following equations:

$$e_{12} = -e_{21},\ e_{22} = -e_{31},\ e_{32} = -e_{41},\ \ldots\ e_{(n-1)2} = -e_{n1},$$
$$P_{12} = P_{21},\ P_{22} = P_{31},\ P_{32} = P_{41},\ \ldots\ P_{(n-1)2} = P_{n1}.$$

We will now make use of formulæ (1) and (2*a*), and begin with the *n*th jar. The formulæ can be used for each jar without further change, if we place before the suffixes 1 and 2 the figure which indicates the position of the jar in the series. We therefore obtain:

	$P_{n2} = 0$
$e_{n1} = -e_{n2}$	$P_{n1} = \frac{d}{ab} e_{n2}$
$e_{(n-1)2} = -e_{n1} = e_{n2}$	$P_{(n-1)2} = \frac{d}{ab} e_{n2}$
$e_{(n-1)1} = -\left(1 + \frac{d}{a}\right) e_{n2}$	$P_{(n-1)1} = 2\frac{d}{ab} e_{n2}$
$e_{(n-2)2} = \left(1 + \frac{d}{a}\right) e_{n2}$	$P_{(n-2)2} = 2\frac{d}{ab} e_{n2}$
$e_{(n-2)1} = -\left(1 + 3\frac{d}{a}\right) e_{n2}$	$P_{(n-2)1} = 3\frac{d}{ab} e_{n2}$
$e_{(n-3)2} = \left(1 + 3\frac{d}{a}\right) e_{n2}$	$P_{(n-3)2} = 3\frac{d}{ab} e_{n2}$
$e_{(n-3)1} = -\left(1 + 6\frac{d}{a}\right) e_{n2}$	$P_{(n-3)1} = 4\frac{d}{ab} e_{n2}$
$e_{(n-4)2} = \left(1 + 6\frac{d}{a}\right) e_{n2}$	$P_{(n-4)2} = 4\frac{d}{ab} e_{n2}$
$e_{(n-4)1} = -\left(1 + 10\frac{d}{a}\right) e_{n2}$	$P_{(n-4)1} = 5\frac{d}{ab} e_{n2}$
. . . . .	. . . .

As we see the potential values proceed according to

the law of the natural progression of numbers, 1, 2, 3, . . ., the charges on the other hand proceed according to the progression

$$1,\ 1+\frac{d}{a},\ 1+3\frac{d}{a},\ 1+6\frac{d}{a},\ 1+10\frac{d}{a},\ \ldots$$

whose differences (differences of the terms) follow the progression

$$\frac{d}{a},\quad 2\frac{d}{a},\quad 3\frac{d}{a},\quad 4\frac{d}{a},\ \ldots$$

We therefore obtain the nth term of the first progression if we add to 1 the sum of the first $(n-1)$ terms of the second progression, or, since this is an arithmetical progression whose difference is $\frac{d}{a}$,

$$1+\frac{n-1}{2}\left(\frac{d}{a}+(n-1)\frac{d}{a}\right),$$

or

$$1+\frac{n(n-1)}{2}\cdot\frac{d}{a}.$$

From this there follows, for the inner spherical shell of the first jar,

$$e_{11}=-\left(1+\frac{n(n-1)}{2}\cdot\frac{d}{a}\right)e_{n2},\qquad P_n=n\frac{d}{ab}e_{n2}.$$

Now since e_{11} is given, we find

$$e_{n2}=-\frac{e_{11}}{1+\frac{n(n-1)}{2}\frac{d}{a}},$$

and therewith the charges and potential values of all spherical shells.

The difference of potential between the inner spherical shell of the first jar and the outer spherical shell of the nth jar is

$$\frac{n\frac{d}{ab}e_{11}}{1+\frac{n(n-1)}{2}\frac{d}{a}}.$$

This difference of potential is, in general, considerably greater than the potential difference in the case of a common battery which, with a charge e_{11} on the inner spherical shell of each jar, would have the magnitude

$$\frac{d}{ab}e_{11};$$

it is, as is seen, $\frac{n}{1+\frac{n(n-1)}{2}\frac{d}{a}}$ times as great. This number is, for a given number of jars, the greater the smaller the ratio $\frac{d}{a}$, *i.e.* the ratio of the thickness of the insulating layer intervening between the spherical shells to the radius of the inner spherical shell. If, however, we choose the number of jars n, so that

$$n=\frac{2a}{d},$$

then is

$$\frac{n}{1+\frac{n(n-1)}{2}\frac{d}{a}}=1.$$

We here gain by the Franklin battery no greater potential difference than if we had taken the first jar alone and had charged it with the same quantity of electricity. If, for example, $a=10$ and $d=0.5$, then is $n=40$. We cannot therefore raise the difference of potential to an unlimited extent by increasing the number of the jars. If the charge of the inner spherical

shell of the first jar always remain the same, the potential difference attains its greatest value when the number of jars n becomes equal to

$$\sqrt{\frac{2a}{d}}.$$

If n exceeds this number, then the potential difference becomes always smaller, and would, if infinitely many jars were connected, entirely disappear. In the particular case $a=10$, $d=0.5$, the maximum potential difference would occur with six jars; the difference of potential would then be 3.43 times as great as if the first jar only had been taken and had been charged with the same quantity of electricity.

The potential difference of a Franklin battery increases in proportion to the charge e_{11} of the inner spherical shell of the first jar. In order to obtain this charge, we must connect the spherical shell with the conductor of an electrical machine and raise this to the potential

$$-\frac{n}{1+\frac{n(n-1)}{2}\frac{d}{a}}\cdot\frac{d}{ab}e_{11}.$$

If we have selected the number n, so that the potential difference in the case of the given charge e_{11} acquires the greatest obtainable value, this will, in general, be considerably greater than the potential to which a single jar or a common battery will be raised by the same charge on the various inner spherical shells, or the potential to which a conductor is brought in the two latter cases, namely,

$$-\frac{d}{ab}e_{11},$$

is considerably smaller than in the first case.

In consequence of the high potential, the charge of the Franklin battery is liable to great outward loss; the arrangement is therefore attended with difficulties. This defect can be remedied by the use of Mach's commutator, whose action consists in converting a common battery into a Franklin battery (Fig. 47). If the inner spherical shell of the first jar of a Franklin battery is to be charged with the quantity of electricity e_{11}, the jars are first connected to form a common battery,

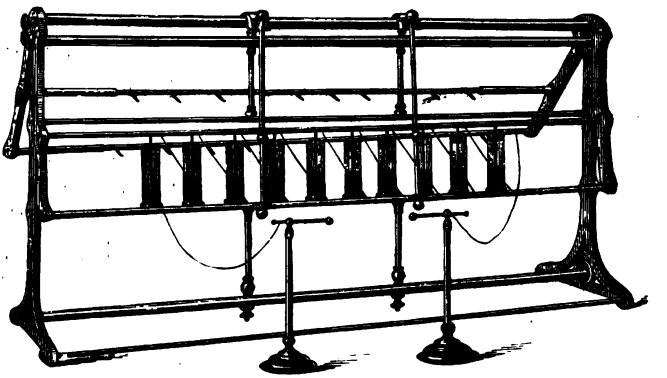

Fig. 47.

each of the jars is charged with the quantity of electricity e_{11}, and after this, by means of the commutator, the Franklin battery is re-established.*

The above determination of the relations in the case of a Franklin battery on the basis of equations (1) and (2*a*) depends on the assumption that the jars do not mutually affect one another; for the equations (1) and (2*a*) were deduced under the assumption that the potential values on the spherical shells are solely and

* The apparatus delineated in Fig. 47 was constructed after the design of Professor Mach by Mr. Franz Hajek, mechanician to the physical laboratory of the German University at Prague.

only determined by the charges thereon. In reality, however, the assumption made is not correct; for example, the potential of the first jar—a uniform distribution of the charge being assumed—at a point distant r from the centre has the magnitude

$$\frac{(n-1)\frac{d}{a}}{1+\frac{n(n-1)}{2}\frac{d}{a}} \cdot \frac{e_{11}}{r}.$$

The potential of a jar thus depends not only on its own charges, but also on the potential values which are produced by the other jars. These latter potential values can be neglected only if the distance of the jars is taken very great.

Under the supposition that the distance between the jars is great, we will now further determine the amount of the energy or the value of the work of the charge of a Franklin battery, or its potential on itself. We obtain this if we multiply the potential value of every spherical shell by half the charge and add the products. We have besides to keep in view the following: the outer spherical shell of each jar is connected with the inner of the jar next following to form a conductor. The potential is on this everywhere equal; the charges, however, being produced by influence, are equal and opposite; consequently the contribution of such a conductor to the total energy of the charge of the battery is zero. If we besides consider that the potential of the outer spherical shell of the last jar is zero, then the contribution of this to the energy of the charge is likewise zero; hence we obtain for the total energy the quantity

$$\tfrac{1}{2} P_{11} e_{11},$$

or
$$-\tfrac{1}{2}\frac{n\frac{d}{ab},}{1+\frac{n(n-1)}{2}\frac{d}{a}}e^2_{11}.$$

The value of the work of a common battery of n jars, each of which is charged with the quantity of electricity e_{11}, has the magnitude

$$-\tfrac{1}{2}\cdot n\frac{d}{ab}e^2_{11};$$

the two amounts of work are related accordingly to one another as

$$1 : 1+\frac{n(n-1)}{2}\frac{d}{a},$$

and the quotients of energy by difference of potential as

$$1 : n.$$

SECTION 38.—Continuation: A Particular Case of the Connection of two Jars.

Let two jars be charged so that the quantity e is on the inner spherical shell of the first, and the quantity $-e$ on the inner spherical shell of the second. Further, let the outer spherical shells be connected together. Separating forces assert themselves at the various points of the outer spherical shells. Negative influence-electricity of the first kind will collect on the outer spherical shell of the first jar, and positive influence-electricity on the outer spherical shell of the second jar. Let the amounts be $-e'$ and $+e'$. These are equal to one another because they are produced by separation. If we make the assumption that the jars are far apart, we can neglect any mutual effect and consider the distribution of the charges as uniform. In this manner we

obtain for the potential on the inner spherical shell of the first jar

$$-\frac{e}{a}+\frac{e'}{b},$$

on the outer spherical shell of the same jar

$$-\frac{e}{b}+\frac{e'}{b},$$

on the inner spherical shell of the second jar

$$\frac{e}{a}-\frac{e'}{b},$$

on the corresponding outer spherical shell

$$\frac{e}{b}-\frac{e'}{b}.$$

Now since both outer spherical shells are formed into one conductor, the same potential must occur on both, or the equation

$$-\frac{e}{b}+\frac{e'}{b}=\frac{e}{b}-\frac{e'}{b}$$

must be true, from which

$$e'=e$$

follows. *The quantities of electricity of the outer spherical shells are therefore equal and opposite to those of the inner spherical shells. The potential of the outer spherical shells is equal to zero.*

SECTION 39.—Two Infinite Parallel Planes.

Let the inner of two concentric spherical shells with centre O (Fig. 48) be charged with the quantity of electricity e and the outer be connected with the earth. The magnitude of the charge on the latter is $-e$. In

consequence of the uniform distribution the density on the inner spherical shell is equal to

$$\frac{e}{4\pi a^2},$$

and on the outer spherical shell equal to

$$-\frac{e}{4\pi b^2}.$$

If we put $b=a+d$, we can write the expression for

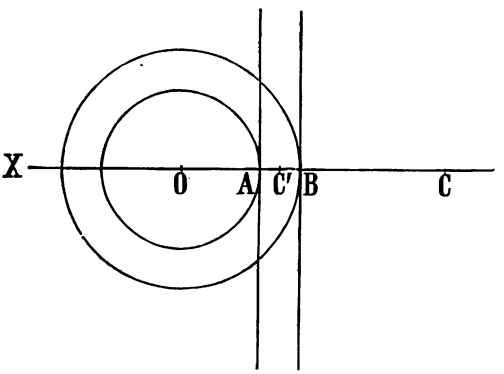

Fig. 48.

the latter density in the form

$$-\frac{1}{1+2\frac{d}{a}+\frac{d^2}{a^2}}\cdot\frac{e}{4\pi a^2},$$

or if we denote the density on the inner spherical shell by σ, in the form

$$-\frac{-\sigma}{1+2\frac{d}{a}+\frac{d^2}{a^2}}.$$

Since the potential at all external points possesses zero value, the electrical force at these points is equal to zero. At all the interior points of the inner spherical

shell the force is also equal to zero. It is only different from zero in the space between the two spherical shells.

Let us consider an external point C and join this with the centre by a straight line. And let us now imagine the points A and B, where the shells are intersected by the straight line OC, to remain fixed, while the point O gradually moves to infinity along the straight line OC in the direction OX. The two spherical surfaces will, by this means, become always greater and greater, and their curvature at the points A and B always smaller and smaller, and finally, when O has arrived at an infinite distance, the curvature equals zero.

When this has happened we can consider the spherical surfaces which pass through A and B as planes. If we take care that on the inner spherical surface the density σ always exists, and that the outer spherical surface always remains in connection with the earth, then the density of the outer spherical surface, when O is at an infinite distance, and therefore a has become infinitely great, and consequently $\frac{d}{a}$ infinitely small, will have the value $-\sigma$.

The electrical force during the conversion of the two spherical surfaces into planes, both outside the exterior spherical surface going through B and inside the interior spherical surface passing through A, will always have zero value, because the laws which we have deduced for concentric spherical surfaces are applicable to every radius, however great. When the spherical surfaces, therefore, at the points A and B appear as planes, outside these planes no electric forces make their appearance.

We will now consider the effect of each of the planes alone. The plane which possesses the density $+\sigma$

repels the unit of positive electricity, if we bring this to a point in the straight line AC, in the direction AC, because everything is symmetrical round the straight line AC. The plane, however, which possesses the density $-\sigma$ attracts the unit of positive electricity—again on account of the symmetry round AC—in the direction CA. The forces which proceed from the two planes are therefore opposite in direction. The resultant of these electrical forces is now, however, at all points of the straight line AC outside A and B equal to zero. The forces therefore proceeding from these planes possess the same magnitude. The various points of the straight line AC have different distances from the two planes, and hence the ratio of these distances changes from point to point. The forces due to the two planes must, however, be everywhere equal in magnitude. It follows from this that the force of each plane *is independent of the distance of the point affected. The plane with the density* σ *produces at each point of the straight line* AC *the same force, and likewise the plane with the density* $-\sigma$ *produces at every point of the straight line the same electrical force. The two forces are equal and opposite to one another.*

Let us now return again to the concentric spherical surfaces. If we bring to the point A the unit of positive electricity, this experiences from the outer spherical shell no electrical force; on the other hand it is repelled by the interior with the force $\frac{e}{a^2}$ or $\frac{4\pi}{4\pi a^2}.e$ or $4\pi\sigma$. If we now, as previously, cause the point O to move to infinity and take care that the density σ always exists on the inner sphere, the force $4\pi\sigma$ will always act on the unit of positive electricity in the direction AC, and therefore also when the point O has arrived at infinity

and the spherical surfaces at A and B have assumed the form of planes. We may now, in this case, suppose the force $4\pi\sigma$ to proceed from both planes. The plane with the density σ tends to repel the unit of electricity in the direction AC; the plane with the density $-\sigma$, on the other hand, tends to attract it in the direction AC. Both planes therefore act with forces directed in the same manner, or the forces of both are combined and give $4\pi\sigma$ as their resultant. Since, as has been before shown, the forces possess the same magnitude, *$2\pi\sigma$ must be the force with which an infinite plane with the density σ repels the unit of positive electricity, and $-2\pi\sigma$ must also be the force with which an infinite plane of the density $-\sigma$ attracts that unit.*

We will now somewhat generalise the investigation of two infinite parallel planes. Let two infinite parallel planes be given at a distance c apart, and let them be charged uniformly with electricity such that on the unit surface of the one the quantity σ and on the unit surface of the other the quantity σ' occurs. We can again make use of Fig. 48 for illustration.

We will for the sake of simplicity designate the two infinite planes which pass through A and B as "plane I" and "plane II." If the unit of positive electricity is brought from infinity along the straight line AC, the electrical forces of plane I thereby perform work, and this work is nothing else than the potential due to plane I at the point A. We obtain this work if we multiply the magnitude of the constant force by the space passed over, and prefix to the product the positive or negative sign according as the direction of the force is coincident with the direction of the motion or the converse. If we denote the potential of plane I at the point A by A and at the point C by C, then the poten-

tial difference A — C is the work which is performed by the displacement of the unit of positive electricity from C to A. This is equal $-2\pi\sigma \times AC$. It is positive or negative according as σ is negative or positive.

The same considerations exactly are applicable to plane II; we need only substitute σ' for σ. If we denote the potential of plane II at the point B by B, then the potential of plane II at the point A equals

$$B-(-2\pi\sigma'.c), \quad \text{or } B+2\pi\sigma'.c.$$

The potential of plane I at the point B is equal to

$$A-(-2\pi\sigma.c), \quad \text{or } A+2\pi\sigma.c.$$

If we denote the total potential at the point A by V_1 that at the point B by V_2, the equations

$$V_1 = A + B + 2\pi\sigma'.c,$$

and

$$V_2 = A + B + 2\pi\sigma.c$$

result.

At a point C', which is distant x from A, the potential of plane I is equal

$$A + 2\pi\sigma.x,$$

the potential of plane II is equal

$$B + 2\pi\sigma'.(c-x),$$

therefore the total potential—we will call it V—is equal to

$$V = A + B + 2\pi\sigma'.c + (2\pi\sigma - 2\pi\sigma')x,$$

or, if we make use of the relations of the former equations,

$$2\pi\sigma - 2\pi\sigma' = \frac{V_2 - V_1}{c},$$

$$V = V_1 + \frac{V_2 - V_1}{c}x.$$

i.e. *the potential follows, at the points of the straight line* AC *between the two planes, the law of a straight line.*

The force by which the unit of positive electricity at the point C' is driven in the direction AB is equal

$$2\pi\sigma - 2\pi\sigma' = \frac{V_2 - V_1}{c}.$$

This force is at all points of the straight line AB (*between the two planes*) *of the same magnitude and direction.*

From the last equation, if the quantities σ, σ' and c are given, the potential difference $V_2 - V_1$ is found. If we wished to determine the potential difference in this way, it would be found to be impracticable. A method of determination is founded on the following observation. In plane I let a portion of the surface be bounded by S units of surface, and likewise in plane II, opposite the surface S, let a precisely equal portion of surface be marked off. If we assume that $\sigma' = -\sigma$, the equation

$$\sigma = \frac{V_2 - V_1}{4\pi c}$$

follows, and therefore the quantity of electricity on the first portion of surface becomes

$$E = \frac{V_2 - V_1}{4\pi c} \cdot S,$$

and that on the second portion becomes

$$E' = -\frac{V_2 - V'}{4\pi c} \cdot S.$$

Let us now investigate the energy or the value of the work of the charges E and E'. We obtain this value of work if we for both portions of surface take half the products of the quantity of electricity and the potential and add these: thus

$$\tfrac{1}{2}(V_1 E + V_2 E'),$$

or

$$\tfrac{1}{2}(V_1 - V_2)E,$$

or if we put

$$V_2 - V_1 = 4\pi c \cdot \frac{E}{S},$$

$$-2\pi c\frac{E^2}{S}.$$

If we imagine the two planes to be displaced by a force in such a way as to remain parallel, and that the distance acquires the magnitude c', then the energy will become equal to

$$-2\pi c'\frac{E^2}{S}.$$

The difference of the two amounts of energy, or

$$-2\pi(c'-c)\frac{E^2}{S},$$

represents the work which the electrical forces, acting between the two planes on the one hand and the portions of area S on the other, perform during the displacement. If c' is greater than c, the direction of the displacement is then opposite to the direction of the force, and the work is negative. The magnitude of the displacement is $c'-c$. If we divide the absolute value of the work by this displacement, we obtain the magnitude of the force by which both portions of surface are attracted. If we call this F, then

$$F=2\pi\frac{E^2}{S},$$

or

$$E=\sqrt{\frac{FS}{2\pi}},$$

or

$$V_2-V_1=\frac{4\pi c}{S}\sqrt{\frac{FS}{2\pi}}=c\sqrt{\frac{8\pi F}{S}}.$$

If we know the force F, the area S, and the distance c, we also know the potential difference V_2-V_1. We shall see later how this method is employed for the determination of potential differences.

SECTION 40.—**Electrical Capacity of a Conductor.**

Let a conductor L (Fig. 49) of any size and shape whatever be charged with a certain quantity of electricity, and be infinitely distant from all other electrified masses. The quantity of electricity, E, undergoes such a distribution that the potential at all points of the conductor possesses one and the same value; the density, however, will in general be different from point to point. If we imagine the unit of positive electricity brought from infinity to the point C, the work done by electrical force during the process will be the potential at the point C. If we consider a very small area on the surface of the conductor, and if we name the quantity of electricity found there *e*, then the force with which the unit of positive electricity acts on this at the point C is equal $\frac{e}{r^2}$, where r denotes the distance of the unit. The force is proportional to the quantity *e*; if this becomes 2, 3, 4, . . . times as great, so will also the force. If we now imagine a still further quantity of electricity of the magnitude E brought to the conductor L, this must also be distributed in the same manner as the first quantity, in order that the potential at all points of the conductor may acquire the same value, and hence the density at the various points will be doubled. At the portion of the surface considered, the quantity of electricity acquires the magnitude 2*e*, and this holds true of all the other portions of the surface of the conductor. The force with which the electricity of the conductor attracts the unit of positive electricity is now double what it was before, wherever

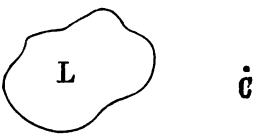

Fig. 49.

this may be situated; the work which is done, owing to the motion of that unit from infinity to the point C, *i.e.* the potential at the point C, will therefore also be double as great.

In exactly the same manner we can show that if the charges of L become equal to 3E, 4E, . . . the potential will be 3, 4, . . . times as great. Since the point C might be chosen anywhere, then what has been stated is true of every point of space, and therefore of the points of the conductor L itself. At these points the same potential everywhere prevails, viz. the potential of the conductor; this increases accordingly proportionally to the charge.

If we (Fig. 50) draw from the point O two straight lines OX and OY, which are at right angles to each other, and mark off along OX spaces representing the charges at successive instants, and erect at the ends of these spaces perpendiculars equal to the corresponding values of the potential of the conductor, the ends of these perpendiculars will be situated on a straight line OM, which passes through the point O, because at O the potential of the charge corresponds to zero. The potential here stands in a similar relation to the charge as, for example, the pressure of a liquid against the bottom of a cylindrical vessel to the quantity of the liquid, or as the temperature of a body to the quantity of heat which is necessary to raise the body from 0° to the temperature observed. A quantity OA corresponds to the potential AB. The

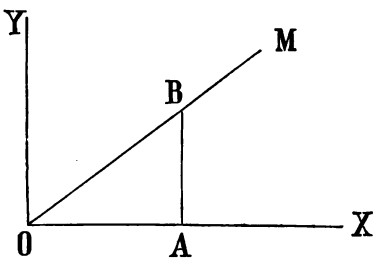

Fig. 50.

energy of the conductor is given by half the product of the potential and the quantity, or by $\frac{1}{2}$ OA × AB, *i.e.* by the area of the triangle OAB.

If we have an insulated conductor and several other conductors in the neighbourhood connected with the earth, the first conductor, as soon as it is charged with a certain quantity of electricity, will act by influence on the other conductors. If L is charged positively, the influence-electricity on the other conductors is negative. The potential of the conductors connected with the earth is equal to zero. The potential on the conductor L—we will denote it by P—is not only determined by the quantity E, but also by the negative influence-electricity of the other conductors. The share which is derived from this is of the opposite sign to that which is derived from E. If the quantity E is doubled, the electrical forces which proceed from the charge of the conductor L will become double as great, and the influence-electricities on the other conductors will become double as great. We can represent the case in such a manner as to imagine the new charges superposed on the old in a completely similar arrangement, viz. in such a manner that the densities at the several points of the conductor are now double of the former densities. The electrical forces are at all points in space the same in direction as before, but possess double strength; hence it follows that the potential at all points of space, therefore also on L, acquires double value.

If we charge L with 3E, 4E, . . . the potential will become 3, 4, . . . times as great, or the potential on L *increases proportionally to the charge.* If we again draw (Fig. 51) two straight lines OX and OY, at right angles to one another at O, and represent on the straight line OX the actual quantities of electricity

which are found on L, and erect at the ends of these spaces perpendiculars whose lengths represent the potential values corresponding to the charges, then the terminal points of these perpendiculars lie in a straight line OM, which passes through O, because when the conductor L is uncharged the other conductors also have no charge, consequently the potential on L is equal to zero.

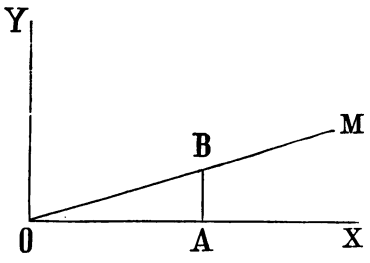

Fig. 51.

If OA represents the quantity of electricity on L, then AB is the potential corresponding to this. The ratio $\frac{OA}{AB}$, *i.e.* the ratio of the quantity of electricity to the potential, is the same for all points of the straight line OM, and therefore the same for all charges of the conductor L. The charge and the potential on L are, as can be easily seen, of opposite sign, the ratio is therefore negative. If we put

$$\frac{OA}{AB} = -m,$$

we may call the quantity m the *electrical capacity of the conductor* L.* If $AB = -1$, then $OA = m$, the capacity is measured by that quantity of positive electricity which raises the conductor to the potential -1, when all the conductors in the neighbourhood are kept at the potential zero.

Not only has the magnitude of the charge on the conductor L effect on the potential of this conductor,

* The notion "electric capacity" could be expressed in German by "*elektrisches Fassungsvermögen*" (containing power).

but the distribution of this charge and the magnitude and distribution of the influence-electricity have also effect. It follows from this that the potential, and consequently the capacity of the conductor L, really depend on the magnitude and shape of this conductor and on the position, magnitude, and shape of the other conductors. The nearer the latter conductors stand, and the greater the influence-electricities, the smaller the absolute value of the potential on L, and therefore the greater is the capacity of L.

If we have to deal with a sphere outside of which no conductor is situated within finite distance, the potential of the sphere will be determined solely and entirely by its own charge. If this is equal to e, the potential on the sphere is equal to $-\frac{e}{a}$, where a denotes its radius.

The capacity is therefore equal to

$$-\left(e \div \frac{e}{a}\right) = a,$$

i.e. equal to the radius. A sphere of 100 cm. radius will need 100 times as much positive electricity to be raised to the potential -1 as a sphere of 1 cm. radius. The capacity of the earth is 636,620,000, *i.e.* 636,620,000 units of positive electricity are necessary in order to raise the earth to the potential -1.

If several spheres whose radii are to each other as $1 : 2 : 3 : 4, \ldots$ are connected together so as to be far apart, and if we charge this system to -1 potential, then the quantities of electricity on them are as the radii, or as $1 : 2 : 3 : 4 \ldots$

If we divide the charge of each sphere by the area of its surface, we obtain the density of the charge. Now, since the charges are proportional to the radii, but the

areas of the surfaces to the squares of the radii, *the densities are therefore inversely proportional to the radii.*

If the radius of one of these spheres is diminished, the density on it is increased; but this does not go on without limit, for experience shows that if the surface density of a sphere exceeds a certain limit, the electricity escapes in the form of a brush. This can be easily explained. The density is in reality proportional to the electrical force. This tends to repel electricity from the conductor in the direction of the normals; this, however, is prevented by the insulator. If, however, the insulator is not perfect, when the force and therefore the density also have acquired a certain magnitude, the electricity will follow the impulse and flow outwards. If the system of spheres therefore contains a very small sphere, the potential does not need to attain any particularly high value for an outward current to take place and last until the density on the small sphere has sunk below the limit stated. The smaller the sphere is, the more complete is the discharge of the system. This explains the ascertained effect of points.*

If a conductor which is charged with the unit of positive electricity aquires the potential -1, it possesses the *unit of capacity.* If we carry a sphere of 1 cm. radius to a great distance from other electrified masses, then this has the unit of capacity. For under these circumstances the capacity of the sphere is equal to the radius, therefore equal to the unit of length.

* When electrified points are spoken of, *i.e.* points at which a finite mass of electricity is concentrated, by these are always meant infinitely small spheres, whose centres coincide with the points in question. This conception is, however, in consequence of what has been stated above, merely a mathematical fiction which can never be realised in nature. It is, however, employed, because by it the laws of the phenomena can be more easily explained.

Two conductors which mutually present a large surface, and are at a small distance apart, are usually named a condenser. Such, for example, is the Leyden jar, Franklin's plate, Kohlrausch's condenser, etc. If in using one of these instruments one of the conductors is connected with the earth, then the other must be charged with a great quantity of electricity in order to acquire the potential -1; it, in this case, therefore possesses a great capacity.

The potential of the inner spherical shell of the Leyden jar, when the outer is connected with the earth, is equal to $-\frac{d}{ab}e$, where a and b denote the radii and e the charge, and $d=b-a$. In conformity with this we obtain for the capacity the expression

$$\frac{ab}{d}.$$

The capacity is greater the smaller the thickness of the insulating intervening layer.

If the inner spherical shell is connected with the earth and the outer charged with the quantity e, the potential on the latter is then equal to $-\frac{d}{b^2}e$; therefore the capacity is equal to

$$\frac{b^2}{d}.$$

The ratio of this is to that in the first case as

$$b : a.$$

If we have n equal jars connected abreast, the outer spherical shells connected with the earth, and each one of the inner spherical shells charged with the quantity of electricity e, the potential is then equal to $-\frac{d}{ab}e$. The total quantity of electricity on the inner spherical

shells of the battery is ne; the capacity therefore of each of such batteries is equal to

$$\frac{nab}{d}.$$

It is n times as great as that of one jar. If we have n equal jars connected so as to form a Franklin battery, and if the inner spherical shell of the first jar is charged with the quantity of electricity e, we produce in that case the potential,

$$-\frac{n}{1+n\frac{(n-1)}{2}\frac{d}{a}}\frac{d}{ab}e.$$

The capacity of these spherical shells is therefore equal to

$$\frac{1+n\frac{(n-1)}{2}\frac{d}{a}}{n}\cdot\frac{ab}{d},$$

or

$$\frac{1}{n}\cdot\frac{ab}{d}+\frac{n-1}{2}b;$$

the capacity in this case can be considered as consisting of two parts. The first part is equal to $\frac{1}{n}$th of the capacity of a single jar whose outer spherical shell is connected with the earth, the second part is equal to the capacity of a sphere whose radius is $\frac{n-1}{2}$ times as great as the radius of the outer spherical shell.

If $a = 10$ cm. and $b = 10.5$ cm., the capacity of a jar whose outer spherical shell is connected with the earth is equal to

$$\frac{105}{0.5} \text{ or } 210,$$

i.e. it needs the same quantity of positive electricity as

a sphere with radius 210 cm. to become raised to the potential — 1.

The capacity of a jar whose inner coat is connected with the earth is in this particular case

$$\frac{10.5^2}{0.5} \text{ or } 220.5.$$

If we combine five such jars to form a common battery, we obtain a capacity equal to

$$5 \cdot \frac{105}{0.5} \text{ or } 1050,$$

i.e. the battery in this case needs as much positive electricity to become raised to — 1 potential as a sphere with radius 1050 cm.*

If five jars are united so as to form a Franklin battery, the capacity possesses the magnitude

$$\frac{1}{5} \cdot \frac{105}{0.5} + \frac{5-1}{2} \cdot 10.5 \text{ or } 63.$$

Of two parallel planes, if the one has the potential zero and the other the potential V_1, and if we consider on both equal portions of the surface of area S, which stand opposite very near one another, then the quantity of electricity on each portion of the surface which possesses the potential V_1 is equal to

$$-\frac{V_1}{4\pi c} S,$$

where c denotes the distance of the two planes. The capacity in conformity with this possesses the value

$$\frac{S}{4\pi c}.$$

In Kohlrausch's condenser two plane circular plates,

* The earth has the same capacity as a common battery of 6360 jars whose inner spherical shells have a radius of 100 cm., and whose outer spherical shells have a radius of 100.1 cm.

which are separated from one another by a layer of air, stand parallel and opposite one another. In calculating the effect of this it is assumed that the electricity on both plates is as uniformly distributed as on the portions of surface S already considered—an assumption which corresponds the more nearly to the reality the greater the ratio of the radius of the plates is to their distance apart. If r is the radius and c the distance, the capacity is equal to

$$\frac{r^2}{4c}.$$

In what follows the capacities which we have found, and the capacities of a few other conductors and condensers, are grouped together for the sake of distinctness.

Conductors and Condensers.	Capacity.
Sphere (radius r)	r.
Leyden jar: (potential of the outer spherical shell equal to zero)	$\frac{ab}{d}$.
Leyden jar: (potential of the inner spherical shell equal to zero)	$\frac{b^2}{d}$.
n jars connected abreast	$\frac{nab}{d}$.
n jars connected in series	$\frac{1}{n}\frac{ab}{d} + \frac{n-1}{2} b$.
Kohlrausch's condenser (radius r, distance c)	$\frac{r^2}{4c}$.
Cylinder (length l, radius r)	$\frac{l}{2 \log \frac{l}{r}}$.*

* Natural logarithms.

Conductors and Condensers.	Capacity.
Two concentric cylinders (length l, inner radius b, outer radius a)	$\frac{l}{2 \log \frac{a}{b}}$.*
A circular plate (radius r, the thickness neglected)	$\frac{2r}{\pi}$.

The capacity of a conductor depends, as we have seen, not only on its own magnitude and form, but also on the magnitude, form, and position of neighbouring conductors. In addition to the factors stated there is still another, namely, the nature of the insulating intervening layer. If we substitute for instance in Kohlrausch's condenser another gas, or glass, caoutchouc, sulphur, paraffin, shellac, and so on, for air, the capacity will be different.

The insulators therefore possess an influence on the distribution of the electricity on the conductor, and are named for that reason dielectrics. If the condenser is at one time filled with air, at another time with a dielectric, the ratio of the capacity in the latter case to that in the former case is a constant number, which bears the name "dielectric co-efficient," or "specific inductive capacity."

SECTION 41.—**The Discharge of a Conductor.**

If a stone of weight Q falls vertically down from the height h, the gravity of the earth does work to the amount of Qh. During this motion its velocity becomes always greater and greater, so that the increase of its kinetic energy, or, if the stone at the commence-

* Natural logarithms.

ment of its acceleration had no velocity, its actual kinetic energy (expressed by $\frac{1}{2}Mv^2$ or by half the product of the mass M and the square of the velocity) is at each instant equal to the actual work that has been done by gravity. If the height h has been traversed, its kinetic energy is determined by the equation $\frac{1}{2}Mv^2 = Qh$. The whole work done has been converted into kinetic energy.

If we in reality perform this experiment, the stone will come to rest as soon as it has reached the earth's surface. What then becomes of its kinetic energy? If we cause the stone to fall into a water pitcher, then the water will indeed be powerfully agitated, but after a not very long time it as well as the stone will come to rest, and everything will remain at rest, as it was before the motion of the stone. What becomes of the kinetic energy? If we test the water with a thermometer we find that it has become warmer; and, in fact, we find if the water pitcher contains n litres of water at 0° C., and the weight of the stone amounts to one kilogramme, and the height of the fall to 425 metres, the water will be heated to $\frac{1}{n}$ degree C. (it is not here taken into account that the pitcher, the stone, and the surrounding air experience a small rise of temperature, the first by both it and the stone, in consequence of its falling on the bottom of the pitcher, experiencing a concussion, whilst the friction of the air causes some expenditure of kinetic energy; and besides, on account of the rise of temperature of the water and the pitcher, a small quantity of heat passes off by conduction, on the assumption that the temperature of the air is lower). *The work of gravity has therefore been converted into kinetic energy, and this again into heat.*

If we throw the stone upwards, we must impart to it a certain velocity v vertically upwards, by virtue of which it may acquire the kinetic energy $\frac{1}{2}Mv^2$. During this motion the velocity, and along with it the kinetic energy, also diminishes more and more, and finally, when the stone has attained a height h such that

$$Qh = \tfrac{1}{2}Mv^2,$$

the velocity equals zero. Gravity acts during the whole motion in direct opposition to it, and thereby performs a negative quantity of work, viz. $-Q.h$. The kinetic energy has been diminished by the amount $\frac{1}{2}Mv^2$, or, what is the same thing, has increased by $-\frac{1}{2}Mv^2$. If we write the equation in the form

$$-\tfrac{1}{2}Mv^2 = -Q.h,$$

we can say again: the increase of the kinetic energy is equal to the work done. In consequence of its kinetic energy, the stone was in a condition to overcome gravity throughout the entire space h; the kinetic energy had therefore the same effect as if a force had raised the stone to the height h, and thereby had done work, or in other words: *the kinetic energy has become converted into work.*

We find a fitting analogy to these transmutations in common life. A workman performs a definite amount of work and receives for it a definite amount of money. He has therefore given away work and has received money for it, or his work has become converted into money. His employer has received the work, and has given money for it. If the latter transfers the work to a third person he again gets back his money for it. The money which the workman has got for his work he must again part with to pay for his food, his clothing, his lodging, firing, and so on. He may then ask, What has become of my money? It has not dis-

appeared without result, for it has been converted partly into food, partly into clothes, partly into his dwelling-place, partly into fuel, and so on. It is precisely the same with kinetic energy. If this is lost then work appears in its stead, or heat, or chemical energy, or electrical force, or the like. The work which is stored up in gunpowder or dynamite as chemical energy appears immediately as mechanical work when we blast rocks with it. It is also the same as regards electrical energy. If we suppose the outer coat of a Leyden jar to be connected with the earth, and a quantity of electricity to be brought from infinity (or from the earth) to the inner, those portions of electricity which are already on the inner coat will offer resistance to the additional quantities, which are added to make up the charge, by repelling them. This force must be overcome; the amount of the work expended in overcoming it is equal and opposite to the work of the electrical forces. If I have now brought on to the inner coat the electrical quantity e, and by this produced the potential V, then the amount of the work which I have put into the jar is equal and opposite to the product $\frac{1}{2}Ve$, or $-\frac{1}{2}\frac{d}{ab}e^2$.

In charging the jar I have, in order to overcome the electrical forces, expended mechanical work, and this is stored up as electrical energy in the jar. Now, as work can be converted into kinetic energy, heat, chemical energy; and conversely, kinetic energy, heat, chemical energy into work, so also is electrical energy capable of being converted into work or some of its equivalents, such as kinetic energy, heat, chemical energy. And, in fact, if we unite the inner coat of a Leyden jar with the outer by a conducting wire, a

movement of the electricities ensues, the positive electricity, if it had been collected on the inner coat, flowing over to the outer, and the negative electricity flowing to the inner coat, and the motion continues until the same potential, *i.e.* the potential of the earth or the potential zero, has been attained by both coats, which are not only connected with one another to form one conductor, but are also in conducting connection with the earth. The electrical energy is then equal to zero, or the whole stored-up work has disappeared. Where then has it gone?

We have connected the one end of the wire with the outer coat and brought the other nearly close up to the inner coat. As soon as this end had come within a certain distance of the inner coat a discharge followed in the form of a spark. This spark emits a dazzling white light and is accompanied by a sharp report. The light has its origin in the white glow of the air, the sound in the compression of the air. The former effect requires heat, the latter mechanical work. If still further we test the wire, it will prove itself to have been heated. We therefore see that the electrical energy has been converted partly into heat and light, partly into mechanical work.

If a plate of glass is inserted in the closed circuit, this can be penetrated (Lullin's experiment). The aperture in the glass plate appears as if the perforation had proceeded outward from the interior towards both sides. Further, the discharge of a condenser, as is well known, is employed for the purpose of firing mines. If the current of discharge is sent through a chemically compound liquid it produces electrolysis, *i.e.* a decomposition of the liquid into its constituent parts. If, for example, two silver wires are plunged into a solution

of copper vitriol, and one is connected with the positive, the other with the negative electrode of an electric machine, it soon appears that the latter is coated with copper. If the electrodes are changed the first wire will become free from copper, the other will, on the other hand, be covered with copper. Here the electrical energy has been converted into the work of chemical decomposition.

The amount of work of the electrical force that is stored up in a Leyden jar when the outer coat is in connection with the earth and the inner charged with the quantity of electricity e, amounts, as we have seen, to

$$-\tfrac{1}{2}\frac{d}{ab}e^2,$$

or

$$\tfrac{1}{2}\frac{d}{ab}e^2$$

is the work which we have done against electrical force while charging the jar. This becomes 4, 9, 16 . . . times as great when the charge becomes 2, 3, 4 : . . times as great, and further, is greater the greater the thickness d of the insulating layer.

If the jar is discharged, its electrical energy vanishes. The work done by electrical forces during the discharge is, therefore,

$$0-\left(-\tfrac{1}{2}\frac{d}{ab}e^2\right),$$

or

$$\tfrac{1}{2}\frac{d}{ab}e^2.$$

If we bring about the discharge by first removing the connection of the outer coat with the earth and then connecting both coats together, the same quantity of work will become free. For the outer coat, after the removal of its connection with the earth, still retains

the potential zero. There will remain on it, as before, the quantity of electricity $-e$, and now if both coats are connected together, we obtain one conductor on which the quantity of electricity $e-e$ or zero is collected. The energy of the jar is therefore after the discharge equal to zero, or the work become free is the same as before.

If we connect the outer coats of two completely equal jars with the earth and charge the inner coat of the first jar with the quantity of electricity e, then the energy is equal to $-\frac{1}{2}\frac{d}{ab}e^2$. If we now connect the wire W′ (Fig. 52) with the inner coat of the first jar and bring the other end near the inner coat of the second jar, a spark will spring over at a certain distance, and thereby the second jar will have become charged. The quantity of electricity which is now on the two coats has the magnitude e; there occurs therefore on the inner coat of each jar the amount $\frac{e}{2}$, so that the potential on the inner coat of each jar acquires the value $-\frac{1}{2}\frac{d}{ab}e$, and consequently the energy of the two jars is equal to

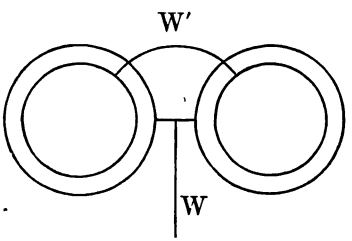

Fig. 52.

$$-\tfrac{1}{2}\cdot\tfrac{1}{2}\frac{d}{ab}\cdot e\cdot e=-\tfrac{1}{4}\frac{d}{ab}e^2.$$

The energy has therefore become changed, viz. to the amount

$$-\tfrac{1}{4}\frac{d}{ab}e^2-\left(-\tfrac{1}{2}\frac{d}{ab}e^2\right)=+\tfrac{1}{4}\frac{d}{ab}e^2.$$

Consequently, $\frac{1}{4}\frac{d}{ab}e^2$ is the work done by the electrical forces during this operation. This has been converted into other forms of energy, viz. in the electrical spark into heat, light, and sound, and a further quantity of heat is generated in the wire W', which exhibits a rise of temperature.

If we have two common batteries, one of which contains m, the other n equal jars, and whose outer coats are not only connected together, but are also connected with the earth, and if we charge the first battery so that each of the inner coats acquires the quantity of electricity e, the total energy amounts to

$$-\frac{m}{2}\frac{d}{ab}e^2.$$

If we then connect the inner coats of the first battery with the inner coats of the second battery, the quantity of electricity me will be distributed over $m+n$ jars in such a way that the inner coat of each of the jars receives the quantity $\frac{m}{m+n}e$. If we now calculate the energy, we obtain

$$-\tfrac{1}{2}\frac{m^2}{m+n}\cdot\frac{d}{ab}e^2.$$

The electrical force, during this process has done work amounting to

$$-\tfrac{1}{2}\frac{m^2}{m+n}\cdot\frac{d}{ab}e^2-\left(-\tfrac{1}{2}m\frac{d}{ab}e^2\right)=+\tfrac{1}{2}\frac{mn}{m+n}\cdot\frac{d}{ab}e^2,$$

which, as before, has been converted into heat, light, and sound.

Riess made a great number of experiments in regard to the development of heat in the connecting wires during the discharge of Leyden jars.

He used for this purpose the air-thermometer invented by him. This in principle (Fig. 53) consists of a glass globe of from 8 to 10 cm. in diameter, to which is affixed a cylindrical rather narrow glass tube of perhaps 450 mm. in length, which widens out at the other end into a wider glass vessel. There is introduced into this tube a liquid which shuts off the air in the globe and in the remaining part of the tube. At the points A and B, which are diametrically opposite one another

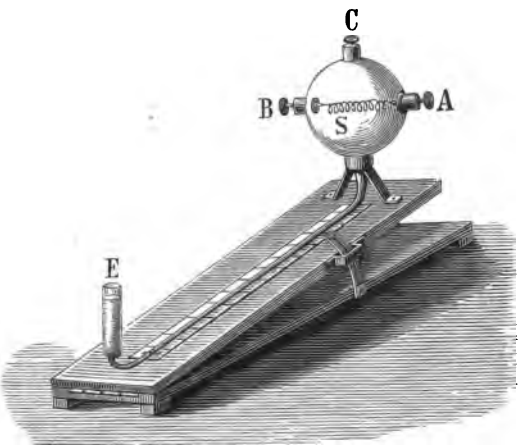

Fig. 53.

the wires which come from the two coats of the jar or battery are inserted. These two points are also connected inside the globe by a thin spiral wire. A third opening C, which can be closed air tight, serves for the regulation of the density of the air in the globe, and consequently determines the position of the liquid in the narrow tube. The glass tube is placed in a position inclined to the horizon. Beneath this there is a scale by which the position of the liquid can be read

off. If now the Leyden jar is discharged through the wire circuit, the wire in the globe becomes heated, its temperature becomes higher than that of the air, in consequence of which heat passes over to the latter by conduction. The heated air expands and pushes back the liquid in the glass tube. The greater the rise of temperature of the wire the greater the rise of temperature of the air; hence the greater its pressure, and the greater is the depression of the liquid. A simple calculation shows that, if the heat which the air gives up to the glass envelope may be neglected, the heat developed in the wire can be put proportional to the number of millimetres through which the liquid is driven back in the tube. We will call this number of millimetres δ. Now Riess has shown that if m jars are connected to form a common battery, and the inner coat of the battery is charged with the quantity of electricity E,—the energy of the battery being then equal to $-\frac{1}{2}m \cdot \frac{d}{ab} E^2$,—the number δ increases in the proportion $1 : 4 : 9 : 16 \ldots$ when E is increased in the proportion $1 : 2 : 3 : 4. \ldots$ If the number of the jars m increases while the total charge E remains the same, then Riess showed likewise that the scale-reading δ decreases in the proportion $1 : \frac{1}{2} : \frac{1}{3}. \ldots$ The quantity of heat developed in the wire therefore increases in the proportion $1 : 4 : 9 : 16 \ldots$ if, while the number of jars remains the same, E increases in the proportion $1 : 2 : 3 : 4, \ldots$; on the other hand it decreases in the proportion $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} \ldots$ if, the quantity of electricity E remaining the same, the number of the jars increase in the proportion $1 : 2 : 3 : 4. \ldots$

If the wire circuit of the Leyden jars consists everywhere of the same metal and the same thickness (cross-

section), then equal lengths will experience an equal rise of temperature. If we insert different wires in the globe, the length and thickness (cross-section) remaining the same, the scale-reading δ depends on the nature of the metal. The material remaining the same, the scale-reading is smaller the greater the length and the thinner the wire (the smaller the cross-section). Vorselmann de Heer deduced from the experiments of Riess that the quantity of heat which is developed in an entire metallic circuit of wire by the discharge of a battery, which is always charged with the same quantity of electricity, is always the same, and that it is directly proportional to the square of the total discharge, but inversely proportional to the number of the jars. As is seen, this experimental result completely agrees with the expression for the energy.

The rise of temperature in the connecting wire must naturally be less when the discharge current performs work, if, for example, it has to penetrate in the form of a spark an insulating layer (air, oil of turpentine, glass, paper, and so on). The greater the amount of this work the smaller is the quantity of heat developed in the wire.

SECTION 42.—**Electrometers.**

If we have in the course of our investigations raised conductors to different potentials, it is absolutely necessary to ascertain the value of these. When this is ascertained, we then know when the capacity of the conductor is given, the charges themselves, because the charge is equal to the negative product of the potential and the capacity. The gold-leaf electroscope, in the form of Fig. 54, is an apparatus which shows that a

conductor has a higher or lower potential than the earth.

A metallic cylinder, placed horizontally, rests on a glass foot, and has directly over this an opening through which a metallic rod, insulated by ebonite from the metallic surface of the cylinder, projects inwards. At one end of the rod there are two gold leaves attached, at the other a sphere. The leaves are usually placed so as to be capable of motion at right angles to the axis of the cylinder. If the opening of the cylinder is closed on both sides by a fine grating of metallic wire, the leaves are then completely enclosed by a metallic envelope. If this envelope is connected with the earth, it is then brought to zero potential. If we besides bring the sphere to zero potential, then all parts of the surface possess the same potential, consequently the same potential must exist in the interior, *i.e.* zero potential; and therefore a charge of the leaves and consequent separation of them cannot occur. If there is a different potential on the sphere whilst the envelope remains in connection with the earth and therefore at zero, there occurs a difference of potential between the leaves and the envelope, in consequence of which the leaves become charged and mutually repel one another. If, for example, a charged body is brought near to the electroscope, the leaves part asunder, because the potential on the sphere is different from zero. The existence of the electrical condition can be known by means of an

Fig. 54.

electroscope, but the potential to which a given conductor is charged cannot be ascertained by it. Electrometers properly so-called serve for this purpose.

The sine-electrometer of Kohlrausch consists in the main of a magnetic needle *ns* (Fig. 55) which is capable of being turned round a metallic axis A. With this axis the metallic arm BB is firmly connected. When both the needle and the arm are unelectrified, the position of the needle is then in the magnetic meridian. Let the angle which the axis of the magnet makes with the line BB be *a*. If we now suppose the axis A connected with a conductor, which is charged to a certain potential, electricity will flow over both to the magnet and the arm, and the needle and the arm will repel one another. Since the arm is fixed, the needle will be turned away from its former position, from the magnetic meridian. As soon, however, as the needle forms an angle with the magnetic meridian, the magnetic force of the earth makes itself felt and tends to turn the needle back. The directive moment of this force is greater the greater the angle through which the needle has been turned from the magnetic meridian, and in fact the moment increases proportionally with the sine of this angle. Therefore in proportion as the needle is turned from the magnetic meridian by the electrical forces of repulsion, the moment of the force of the earth's magnetism will become greater and greater, and will finally become equal to the directive moment of the electrical forces. When this is reached, the needle then comes to

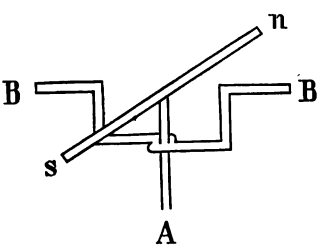

Fig. 55.

rest. Now let us imagine the arm BB turned, viz. in the same direction as the needle was turned. The needle will again be turned in the same manner as before, but the angle between the needle and the arm BB will thereby be smaller. When it has acquired the value α we will then stop turning the arm BB, because the needle has now exactly the same position relatively to the arm as it has before the electrification.

If the angle through which the needle has been turned from the magnetic meridian is ϕ, the directive moment of the magnetic force of the earth is proportional to sin ϕ. This moment is neutralised by that of the electrical forces. The repulsion takes place in such a manner that every point of the arm BB repels every point of the needle. If there is at a point of the arm the quantity of electricity e, and at a point of the needle the quantity e', then the force of repulsion between them will have the magnitude $\frac{ee'}{r^2}$, where r denotes the distance.

If we suppose the experiment just described to be repeated, but in such a way that the density has now at all points double the value it had previously, where the quantities were formerly e and e', they will now be $2e$ and $2e'$, and therefore the force between these points will acquire the magnitude $\frac{2e \cdot 2e'}{r^2} = \frac{4ee'}{r^2}$. If we charge, therefore, the arm and needle with double the quantity of electricity, and turn the arm until the angle between the arm and the needle has again become equal to α, then both have the same relative position as formerly, but the force of repulsion has now become four times as great. If we name the angle through which the needle has been turned from the magnetic meridian ϕ',

then the directive moment of the earth's magnetism is proportional to sin ϕ', and we obtain the proportion

$$\sin \phi' : \sin \phi = 4 : 1.$$

And if the quantities are three, four, fivefold of the former, the repulsion, when the angle between the needle and the arm has again become a, is 9, 16, 25 times as great. The sines of the angles of deviation of the needle must naturally be in the same ratio; *the charges, therefore, of the needle and of the arm are proportional to the square roots of the sines of the angles of deviation.* Since, further, in the case of each charge the same final position of the arm relative to the needle is obtained, namely, that in which it makes with it the angle a, we have in all cases to deal with one and the same apparatus, or the charges on the conductor or in the condenser are proportional to those in the electrometer, or the charges also of the conductor and condenser respectively are proportional to the square roots of the sines of the angles of deviation.

Because arm and needle have always the same relative position, the electrometer has always the same capacity. But the charge is equal to the negative product of the potential and capacity ; it therefore follows that also *the potential values of the conductor and condenser respectively have the same proportion to one another as the square roots of the sines of the angles of deviation.*

The electrometer is represented in Fig. 56. The axis A (Fig. 55), and with it also the arm BB, is in firm connection with the cylindrical case. In the case, which is movable round a vertical axis, is a slit, and diametrically opposite to this a mirror. In this is seen the image of a mark placed above the slit. With the magnetic needle are connected two mirrors, which are inclined a little to the vertical. When the magnetic

needle is at right angles to the straight line joining the slit and the mirror, many rays from the mark fall on the mirror of the case, from this on to the mirror of the

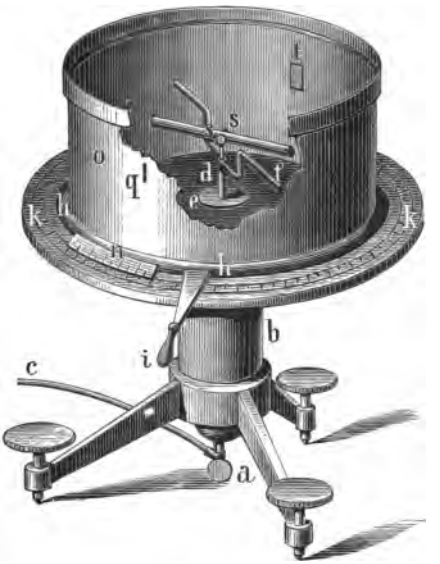

Fig. 56.

magnet, from this back again on to the mirror of the case, and from this through the slit to the eye. The mark is thus not only seen directly in the mirror of the case, but also reflected by the mirror of the magnet.

At the beginning of the experiment the case has such a position that both these images coincide. When the electrometer is charged the case is turned, and along with it the arm until both images cover one another in the mirror. Arm and needle have then the same relative position as before the experiment. The angle through which the case has been turned is equal to the angle of deviation of the needle. This is read off on a

circular scale which is placed at the circumference of the case, and is furnished with a vernier.

The smaller the angle a is, the greater, when the charge remains the same, is the angle ϕ, and conversely. When the charges are small the angle a will be chosen small, when the charges are large a will be made large. Besides, needles with magnetic moments of varying strength can be employed. The stronger the moment is the smaller is the angle ϕ, the charge remaining the same. If strong charges are to be compared with one another, a needle of strong magnetic moment will be selected, and the angle a will be taken very great.

To be able to compare with one another two observations which have been made with two different angles, a_1 and a_2, the following method is adopted. The electrometer is connected with a Leyden jar of great capacity, which is charged with a definite quantity of electricity, and two angles, a_1 and a_2, are chosen. Corresponding to these for ϕ are the two values ϕ_1 and ϕ_2. If we now put the ratio

$$\sqrt{\frac{\sin \phi_1}{\sin \phi_2}} = v,$$

and measure the quantity of electricity e at the angle a_1 and then the quantity of electricity E at the angle a_2, then, if in the first case the angle of the needle read off is ϕ and in the second case ψ,

$$\mathrm{E} = ev\sqrt{\frac{\sin \psi}{\sin \phi}}.$$

The proof is simple. If the quantities e and E engender on the conductor (condenser) the potential values p and P, and if further the potential of the Leyden jar is Π, the quantities of electricity in the electrometer at the angle a_1 are proportional to the potential values, or as

$p : \Pi$. On the other hand, however, the quantities of electricity have the same ratio as $\sqrt{\sin\phi} : \sqrt{\sin\phi_1}$, we therefore obtain the proportion

$$p : \Pi = \sqrt{\sin\phi} : \sqrt{\sin\phi_1}.$$

In precisely the same way there results

$$P : \Pi = \sqrt{\sin\psi} : \sqrt{\sin\phi_2};$$

hence by division

$$\frac{P}{p} = v\sqrt{\frac{\sin\psi}{\sin\phi}}.$$

The potential values P and p are proportional to the electrical quantities E and e; the equation therefore follows,

$$\frac{E}{e} = v\sqrt{\frac{\sin\psi}{\sin\phi}}.$$

The guard-ring electrometer constructed by Sir William Thomson is a very sensitive electrometer. The principle of it is as follows: In section 39 we saw that if two infinite parallel planes stand opposite, and we assume that both are uniformly covered with electricity, viz. the one with positive the other with negative electricity of the same density, two corresponding portions of them which stand directly opposite one another attract one another with a force equal to

$$F = \frac{2\pi E^2}{S},$$

where E represents the quantity of positive electricity on one of the portions and S the area of this. Further, if V_1 and V_2 are the potentials of the two planes, the potential difference will be equal to

$$V_1 - V_2 = c\sqrt{\frac{8\pi F}{S}},$$

where c is the distance between the planes.

If we would practically put this case to the test by placing two circular discs opposite one another in a parallel position, the electricities would be distributed not uniformly, but in such a manner that if we pass on the disc from the centre towards the edge the density continually increases. At the central part, however, the change is not so rapid as near the edge; if we therefore imagine on one of these discs (Fig. 57) a circular area having the same centre but not a large radius, this will turn out to be almost uniformly covered with electricity, and will therefore nearly fulfil the requirement made in relation to the surface S. Naturally no alteration in the conditions is produced if we imagine an incision made along the circular line. The outer circular ring, which forms a continuation of the smaller central disc, and thus brings it about that nearly everywhere on the smaller disc an equal density is produced and maintained, received from Thomson the appropriate name "guard-ring." Thomson's electrometer accordingly consists in reality of two circular plates (Fig. 58), A and B, which possess different potentials. The middle part C of the upper plate is movable, can pass through the guard-ring without friction, but at the same time fits it as closely as possible. The under plate and the guard-ring of the upper one remain fixed. The two plates A and B have the same size and are supported one exactly over the other. The movable disc is hung by a thin metal (platinum) wire on the scale beam L, which can turn round a tightly-stretched metal wire, and is balanced by a weight Q. The metal wire is stretched between two conducting supports which are in con-

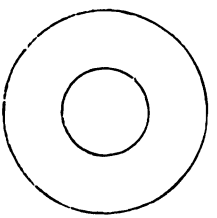

Fig. 57.

ducting connection with the guard-ring. The scale-beam L terminates in a fork which permits a fixed arm to pass between the prongs. On this last are two dark spots, and in front of them a hair fastened, which is stretched in a horizontal direction. If the scale beam moves upwards and downwards this hair, which one can closely observe with the help of a lens *l*, passes over the dark spots. If the under surface of the movable disc and the under surface of the guard-ring are in one and the same plane, the hair lies right in the

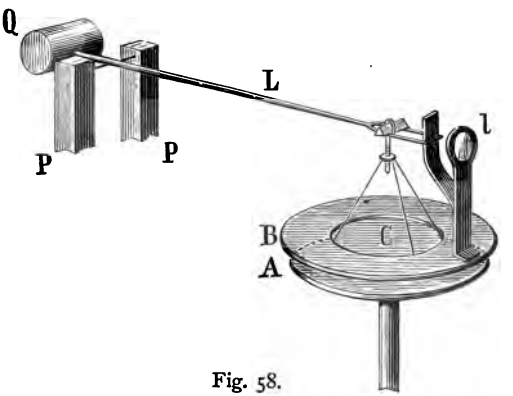

Fig. 58.

middle between the two spots. This position is named the "marked position." The under fixed disc A rests on an insulating support, and can, by means of a micrometer screw, be brought as near as one wishes to the plate B.

When the two plates are not charged, a definite weight must then be placed on the movable disc in order to bring it to the marked position. Let this weight be F. If the potential-difference of two conductors is now to be determined, the one is connected with the under, the other with the upper plate (the guard-ring

and the movable disc are connected by means of the metallic wire by which the latter hangs, the lever arm, the metallic wire round which this turns, the conducting supports PP of this metallic wire, and the conducting arrangement between the supports and the guard-ring), the weight F removed and the plate A brought closer or moved away by means of the micrometer screw until the movable disc C has reached the marked position. The attraction which the disc B now experiences from the under one is equal to the weight F. If now S is the area of the movable plate, c its distance from the plate A, and $V_1 - V_2$ the difference of potential, then from the formula there results—

$$V_1 - V_2 = c\sqrt{\frac{8\pi F}{S}}.$$

The determination of the distance c between the plates is always attended with errors, whose effect is greater the smaller that c is. Thomson has consequently arranged the matter so that at first the plate A, by being connected with the inner coat of a Leyden jar whose outer coat is in connection with the earth or with the pole of a series of cells whose other pole is in contact with the earth, is brought to a constant potential V. The weight F is then removed and the upper plate is connected with the conductor whose potential is V_1. In order to bring the disc C to the marked position, the plate A must be given the distance c_1 from the plate B, and there then results

$$V_1 - V = c_1\sqrt{\frac{8\pi F}{S}}.$$

When this is done, the upper plate B is connected with the second conductor, whose potential is V_2, and the

plate A is given the distance c_2, so as to bring the disc C to the marked position. We now have

$$V_2 - V = c_2 \sqrt{\frac{8\pi F}{S}},$$

and therefore by subtraction

$$V_1 - V_2 = (c_1 - c_2) \sqrt{\frac{8\pi F}{S}}.$$

The difference of the distances $c_1 - c_2$ can be very accurately determined by means of the micrometer screw, and consequently also the difference of potentials $V_1 - V_2$.

This is the most essential feature of the arrangement. During the experiment the potential V must naturally have the same value throughout. To secure this, Thomson constructed two pieces of apparatus, one of which indicates the change of potential, and the other—named the "replenisher"—by increasing or diminishing the charge of the jar keeps the potential at a constant value.

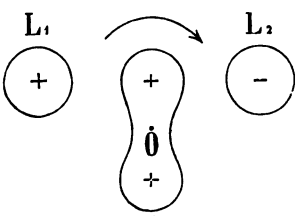

Fig. 59.

The quadrant electrometer, likewise invented by Sir William Thomson, is exceedingly sensitive. We may imagine a conductor A of the form represented by Fig. 59. It consists of a very thin aluminium plate of the shape of the figure 8, which is capable of being turned round the point O, and may be charged with positive electricity. Left and right of this let two conductors L_1 and L_2 be situated in the positions indicated, the former of which may be charged with positive electricity, the latter with negative electricity. L_1 repels both ends

of the plate A, but acts with greater force on the nearer end ; L_2 on the other hand attracts both ends, and also acts more strongly on the nearer end. The result of this is that the plate tends to turn in the way indicated by the arrow, just as a magnetic needle, if one of its poles is brought between the poles of a horse-shoe magnet. If we bring also (Fig. 60) on the two sides of the other end of the aluminium plate two conductors, L'_1 and L'_2, of which the right is charged with positive, the left with negative electricity, they will again act

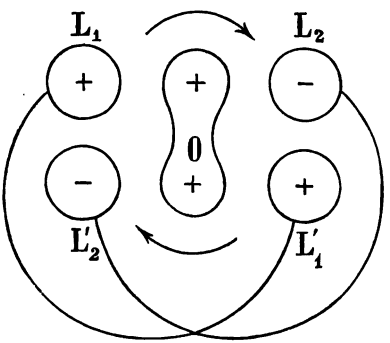

Fig. 60.

most strongly on the ends lying next them, and in the same sense as the other conductors, so that the turning moment will be increased.

If the conductors L_1 and L'_1, L_2 and L'_2 are joined together, and if the conductors L_1, L'_1 have the potential V_1, the conductors L_2, L'_2 the potential V_2, and further, if V is the potential of the aluminium plate, the turning moment, as a closer examination shows, is given by the expression

$$\frac{l V(V_1 - V_2)}{M}\left[1 - \frac{h}{l}\frac{V_1 + V_2}{V}\right]$$

where h and l denote constants depending on the form, the size, and the relative positions of the conductors.

Thomson's quadrant electrometer, in accordance with this principle, is constructed in the following manner. A low cylinder of metal is cut (Fig. 61) in the direction of two diameters at right angles to one another, and is thus divided into four quadrants. The points of the quadrants which would meet at the centre are cut off so as to leave a circular opening. These quadrants represent the conductors L_1, L'_1, L_2, L'_2. The aluminium plate, which weighs about 70 milligrammes and possesses an area of nearly 8 square centimetres, hangs horizontally in the middle of the cylinder, supported by a strong vertical platinum wire. This wire is T-shaped at the upper end, and is suspended from two cocoon threads, which always tend to turn the aluminium plate back to the position of equilibrium as soon as it is brought out of this by electrical forces. The platinum wire also carries a concave mirror S (8 mm. diameter, 22 milligrammes weight, Fig. 62), which throws the image of a flame on a divided scale. When the needle is turned this image moves on the scale, and the amount of the rotation is thereby measured. The lower end of the platinum wire dips into sulphuric acid, which is contained in a rather large vessel of flint glass of the shape of a bell turned upside down (Fig. 63). A piece of platinum, which likewise dips into the sulphuric acid, is fastened to the wire and damps the vibrations of the aluminium plate (needle). The outside of the glass vessel is covered with tinfoil. The glass

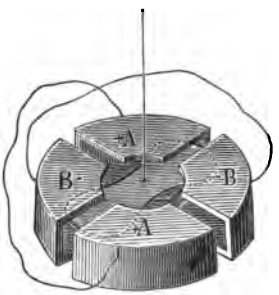

Fig. 61.

vessel thus represents a Leyden jar, whose inner coat is the sulphuric acid and whose outer is the tinfoil. The aluminium plate is in conducting connection with the inner coat: the tinfoil is always connected with the earth. The sulphuric acid offers the advantage that it keeps the air in the interior dry, and thereby ensures complete insulation.

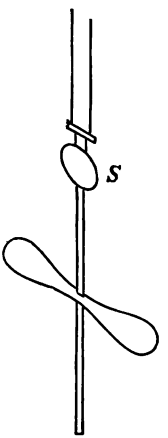

Fig. 62.

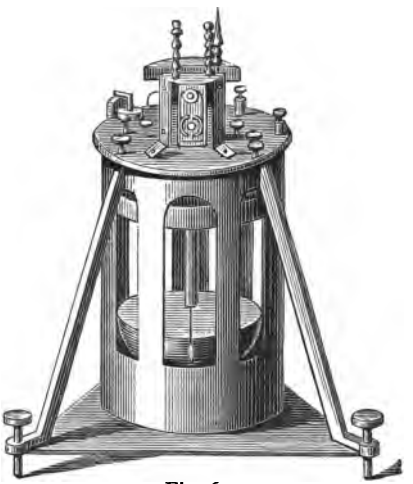

Fig. 63.

The quadrants are, precisely as the conductors L_1, L'_1, L_2, L'_2, connected together crosswise. They are fixed on glass rods and so insulated. In order to keep the sulphuric acid, which forms the inner coat of the Leyden jar, and with it also the aluminium plate, always at the same potential, the same two pieces of apparatus as in the case of the guard-ring electrometer are employed, namely, one which enables a change of potential to be detected, and a second, the replenisher, which by increasing or diminishing the charge keeps the potential constant. In using the apparatus the needle is placed

so that its longitudinal axis coincides with one of the diameters by which the cylinder has been cut into quadrants. Then the sulphuric acid and thereby the aluminium plate is charged to a definite potential V, and this is kept constant with the help of the apparatus mentioned. The two pairs of quadrants are then connected with one another, and that division of the scale is noted on which the concave mirror throws the image of the flame. If the conducting wires of one pair of quadrants are now connected with one, and those of the other pair with the second of the two conductors whose difference of potential is to be determined, the aluminium needle will rotate about the vertical axis until the turning moment of the electrical forces is counterbalanced by that of the two cocoon threads. The image of the flame at the same time moves on the scale through a certain number of divisions towards the right or left. The angle through which the needle turns can be found from the scale-reading and the distance of the scale from the mirror. The moment of rotation which the cocoon threads produce is nearly proportional to the sine of the angle, or if this is very small, to the angle itself. The turning moment of the electrical forces is nearly constant, for as the needle moves away from one quadrant it at the same time approaches to another in such a way that, as the effect of the former becomes smaller, that of the latter becomes greater.

If one pole of a galvanic battery is connected with one pair of quadrants, and the second pole with the other pair of quadrants, then, as we shall see later on, $V_2 = -V_1$, consequently the turning moment is equal to $2lVV_1$.

When we know the quantity lV we know also, because we can easily ascertain the moment due to the

cocoon threads, the potential V_1. If we use a galvanic series of known electromotive force (potential difference), then V_1 is known, and we can from this determine the quantity lV. Let the angle through which the aluminium needle has been turned be ϕ_1. If we take a second galvanic battery whose electromotive force (potential difference), $V'_1 - V'_2$, because $V'_2 = -V'_1$, possesses the value $2V'_1$, the aluminium plate may be turned through the angle ϕ_2. If the potential V has remained the same, and if the two angles ϕ_1 and ϕ_2 are small, the relation is

$$2V_1 : 2V'_1 = \phi_1 : \phi_2,$$

whence

$$2V'_1 = \frac{\phi_2}{\phi_1} \cdot 2V_1$$

is deduced.

The quadrant electrometer is exceedingly sensitive, and is capable of even measuring a difference of potential which does not amount to more than $\frac{1}{700}$ of that of a single Daniell's element.

If, with the help of an electrometer, the potential of a condenser or a conductor is measured, and if the capacity is known, then the quantity of electricity with which it is charged is also known.

In charging a Leyden jar or battery, a small Leyden jar, the so-called Lane's standard jar (Fig. 64) may be used for a rough determination of the amount of the charge. This stands on a conducting bottom plate, which is connected with the earth (or a water or gas pipe). On this, besides, there is a glass rod *g*, on which is placed a small brass rod. This bears at one end a knob *b*, which can be brought as near as one chooses to the knob *a* of the Leyden jar. The outer coat is connected with the small brass rod, the battery is placed on an insulating support, and its outer coat is connected

with the inner coat of the standard jar. If the inner coat of the battery is charged with positive electricity, then the positive influence-electricity passes from the outer coat of the battery to the inner coat of the standard jar, and here acts by influence on its outer coat. The positive influence-electricity flows away from the standard jar to the earth, the negative on the other hand is bound. When the charge of the standard jar, and therefore the difference of potential between the knobs *a* and *b*, has acquired a certain height, a spark passes between the knobs—the standard jar is discharged. Since the distance of the knobs *a* and *b* is fixed, the passage of the spark is always due to the same potential difference, and consequently also to the same charge of the jar. We will designate this "the unit charge." If *n* sparks pass over, the battery has been charged with *n* charges. These quantities are not quite equal to each other, because of a property of Leyden jars which has been designated by the name residual charge. If, for instance, a Leyden jar is discharged by momentarily connecting the two coats, then if, after a certain time, the coats are once more connected together, a second smaller discharge is obtained. In the Leyden jar, therefore, some of the original charge is kept back and produces the second discharge. This quantity is named the residual charge. Owing to this residual charge, the charges which the standard jar obtains between two sparks are not all precisely equal, the first being greater than the succeeding ones.

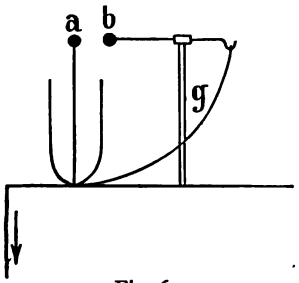

Fig. 64.

CHAPTER III

ELECTRICAL POTENTIAL (ELECTRICAL CURRENT)

SECTION 43.—**Volta's Fundamental Experiment.**

WHEN two insulators, or an insulator and a conductor, are rubbed together, both become charged with equal amounts of opposite electricities. The potential which may before have been zero on both will now be different from zero, and the difference of potential will acquire a value other than zero. The more one rubs the greater this difference of potential becomes, but it is probable that this increase does not continue without limit—that the difference more likely tends towards a definite maximum which cannot be exceeded. The magnitude of the difference of potential under the same circumstances naturally varies with the substances used. It is, indeed, known that one and the same insulator will become positively electrified when rubbed with one insulator and negatively electrified when rubbed with another insulator, and that the quantity of electricity excited in the different cases is different.

The maximum difference of potential for different pairs has not yet been ascertained; but if it were possible to determine this, the insulators could be arranged in a series such that one insulator rubbed with a second standing beneath it in the series becomes

positive, and further, that the maximum potential difference which can be reached is greater the farther apart the insulators stand in the series. Let us now compare with what has been said the following experiment:

A plate of copper is screwed on to a very sensitive electroscope, the copper plate being varnished on the upper side. A varnished zinc plate is placed on this by means of a glass rod. The gold leaves show no tendency to separate. Both plates are now connected by a zinc wire (Fig. 65), which is applied at two points free from varnish, and after this the wire is removed. The gold leaves still show no tendency to separate. If, however, the upper plate (zinc plate) is raised by means of a glass rod, the leaves separate—a proof that the copper plate is charged with electricity. If now a rubbed glass rod is brought near, the leaves come together—a proof that the copper plate has been negatively charged. What is now the cause of this charge?

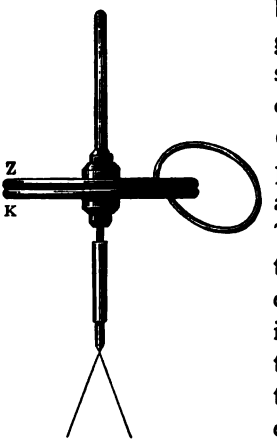

Fig. 65.

No friction has occurred, but merely contact of the zinc wire with both plates. Contact of the zinc wire with the zinc plate has, as one can easily convince himself, no effect in exciting electricity; *the contact of the zinc wire with the copper plate is, therefore, the cause of the excitation.* The copper plate has become negatively, the zinc plate positively electrified. This may be accounted for by saying that the two substances attract the two electricities in a different degree; the

zinc attracts the positive, the copper the negative electricity more strongly, the result being that the positive electricity flows from the copper plate on to the zinc wire, and from this on to the zinc plate, the negative from the zinc plate and the zinc wire on to the copper plate, and in fact this goes on until the separating forces are neutralised by the electrical forces due to the electricities collected acting in an opposite manner. When the motion of the electricities has ceased, then the moving force at all interior points of both plates is zero, or *the potential on each of the plates possesses a constant value.*

In order to be better able to examine these relations, we will consider two clean, polished plates of zinc and copper which mutually touch each other (Fig. 66). The separating forces charge the two plates, the copper plate negatively, the zinc plate positively, until each has attained a constant potential. We will name the potential on the zinc plate V_z, that on the copper plate V_k. Now, as we have previously seen, when two points possess different potentials an electrical force exists whose magnitude, if the points lie very near one another, is given by the quotient of the difference of potentials and the distance of the points. If we now imagine a normal drawn to the surface of contact of the copper and zinc, this will cut the bounding surface of the copper at one point and the bounding surface of the zinc at a second point. Let the distance of the two points, which is infinitely small, be denoted by c. Since the potential difference of these points is $V_z - V_k$, the electric force existing between them due to the accumulated electricities will have the magnitude

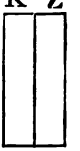

Fig. 66.

$$\frac{V_z - V_k}{c}$$

This force acts at right angles to the surface of contact, just as the force of separation, but in the opposite sense to this, and neutralises it as as soon as the motion of the electricity has ceased. If we name the separating force S, we obtain the equation

$$\frac{V_z - V_k}{c} = S,$$

or

$$V_z - V_k = c.S.$$

The potential difference, as experiment shows, is a finite magnitude different from zero; therefore $c.S$ is also a finite magnitude different from zero. But now, since c is infinitely small, it follows that S is infinitely great. *We must, therefore, represent the separating forces as infinitely great forces.*

We must further assume concerning the separating forces that their seat is essentially at points of the conductor immediately contiguous to the surface of contact, and that the force of separation acting at a spot of the surface of contact proceeds from points situated in the immediate neighbourhood.

The separating forces are severally and only determined by the internal constitution of the substances; when copper and zinc are placed in contact the same separating force must always occur, and therefore *both are brought to the same difference of potential, however great the surface of contact may be.*

We will always designate the product $c.S$ by writing copper|zinc, or $Cu|Zn$, or by an equation

$$V_z - V_k = Zn|Cu.$$

The potential difference $V_z - V_k$ is also named the electromotive force, but by this is understood not a force as formerly, but an amount of work, because potential always signifies work, hence change of potential is a quantity of work. The unit of work, *i.e.* that work

which the unit of force performs when it causes a displacement in its direction to the extent of a unit of length (one cm.), serves as the unit by which we here measure the electromotive force. There exists between two points the unit of potential difference, or the unit of electromotive force, when the electrical force performs the unit of work by the transference of the unit of positive electricity from the one point to the other. $V_z - V_k$ is nothing else but the work which the electrical forces perform by the transference of the unit of positive electricity from a point in the copper plate to a point in the zinc plate, expressed in the units of work above defined. The equation

$$V_z - V_k = Zn|Cu$$

can also be written in the form

$$V_k - V_z = Cu|Zn,$$

so that we obtain the relation

$$Cu|Zn = -Zn|Cu.$$

The quantities of electricity which collect on the two plates are equal and opposite to one another, because both have been produced by separation; their absolute amount, however, depends upon the capacity of the plates. If the plates are of exactly the same shape and size they have then evidently the same capacity, and their potentials must consequently be equal and opposite. The charges of the plates, even when the potentials are very small, may nevertheless be very great if, for instance, the capacities are very great. If the two plates merely touch one another, as in Fig. 66, then the charges are small; if, however, the two plates are made into a condenser, as in Fig. 65, then they will be of considerable magnitude, because the capacity of each plate is then very great. If the two plates are of different shapes and unequal sizes, the difference of potential $V_z - V_k$

will still be the same as before, but the absolute values of the potentials V_z and V_k are different.

We can image the above experiment (Volta's fundamental experiment) modified in the following manner: A zinc and copper plate are brought in contact with one another. On the former the potential V_z occurs, on the latter V_k. Now, if we suppose the two plates connected at the same time with an electrical machine and charged so that, by virtue of the charge alone, the potential V_k would be produced at all points of the pair of plates. The potential of the zinc plate is by this changed from V_z to $V_z - V_k$, while the potential of the copper plate acquires the value zero. As we see, there now exists exactly the same difference of potentials as there was before, but one plate has been brought to zero potential. If the copper plate is now connected with the earth no electricity escapes to the earth, nor will electricity flow from the earth to the copper plate, because no difference of potential exists between the earth and the copper plate. Exactly the same thing must take place if we, without charging the pair of plates with a quantity of electricity, connect the copper plate directly with the earth, for the forces causing separation at the surface of contact remain the same, hence the difference of potentials must also remain the same. The potential of the copper plate is zero, therefore that of the zinc plate must be $V_z - V_k$.

SECTION 44.—**Electromotive Series of Metals.**

Volta found an excitation of electricity not only by contact of zinc and copper, but he found that *all metals exhibit this property.* The difference of potentials is different for different pairs of metals, and the kind of

electricity with which a metal is charged is different according as it is brought in contact with this or that metal. For instance, copper will be negatively electrified by contact with zinc, but positively electrified by contact with silver. This is expressed by saying that copper is "electro-negative" towards zinc, "electro-positive" towards silver. Volta arranged the metals which he investigated in a series, the so-called "electromotive series," which exhibits the following laws:

1. *When a metal of the series is brought into contact with a metal occurring later in the series, it is positively electrified; when brought into contact with one occurring earlier in the series, it is negatively electrified.*

2. *The difference of potential between two metals is greater the greater their distance apart in the series; it is equal to the sum of the differences of potential of the intervening metals.*

The series which Volta established was—zinc, lead, tin, iron, copper, silver, gold. To this he afterwards added carbon, graphite, and manganese.

In regard to law 2, Volta found the following proportionate numbers:

$$\begin{aligned} \text{zinc}|\text{lead} &= 5 & \text{iron}|\text{copper} &= 2 \\ \text{lead}|\text{tin} &= 1 & \text{copper}|\text{silver} &= 1 \\ \text{tin}|\text{iron} &= 3 & \text{zinc}|\text{silver} &= 12 \\ & & \text{tin}|\text{copper} &= 5 \\ & & \text{zinc}|\text{iron} &= 9. \end{aligned}$$

We therefore see that

$$\text{zinc}|\text{silver} = 12 = 5 + 1 + 3 + 2 + 1,$$

or

$$= \text{zinc}|\text{lead} + \text{lead}|\text{tin} + \text{tin}|\text{iron} + \text{iron}|\text{copper} + \text{copper}|\text{silver};$$

further

$$\text{tin}|\text{copper} = 5 = 3 + 2,$$

or

$$= \text{tin}|\text{iron} + \text{iron}|\text{copper};$$

and $$\text{zinc}|\text{iron} = 9 = 5 + 1 + 3$$
$$= \text{zinc}|\text{lead} + \text{lead}|\text{tin} + \text{tin}|\text{iron}.$$

We can illustrate the matter from another point of view. If we connect the metals A, B, C, D, as shown in Fig. 67, there occurs on A the potential V_a, on B

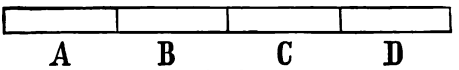

Fig. 67.

the potential V_b, on C the potential V_c, and on D the potential V_d. In accordance with this we obtain the following equations:

$$V_a - V_b = A|B, \quad \text{or } V_b = V_a - A|B,$$
$$V_b - V_c = B|C, \quad \text{or } V_c = V_b - B|C,$$
$$= V_a - A|B - B|C,$$
$$V_c - V_d = C|D, \quad \text{or } V_d = V_c - C|D,$$
$$= V_a - A|B - B|C - C|D;$$

therefore

$$V_a - V_d = A|B + B|C + C|D.$$

Now experiment shows that if we bring the metals A and D directly in contact, the same difference of potential occurs, so that we obtain the equation

$$A|D = A|B + B|C + C|D.$$

If we again add the metal A (Fig. 68) to the series

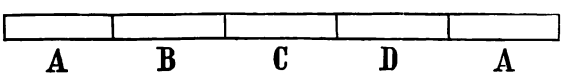

Fig. 68.

of metals A, B, C, D, and if we name its potential $V_{a'}$, the equation $V_d - V_{a'} = D|A$ follows.

But $V_a - V_d = A|D$,

it also follows that

$$V_a - V_{a'} = A|D + D|A = A|D - A|D = 0.$$

If we accordingly have connected together a series of metals, and if we terminate the chain with the same metal with which we begin, the difference of potential between the ends of the chains is then always equal to zero. The laws of the electromotive series have received further confirmation by the experiments of Kohlrausch and Hankel.

The electromotive force between two different metals changes with the temperature, though the law of this change is not yet ascertained.

SECTION 45.—**Metals and Liquids.**

When metals are brought into contact with liquids an excitation of electricity also takes place. The experiment is analogous to the former one. A metal plate K (Fig. 69) is screwed on to a very sensitive electroscope, and a glass plate covered with shellac is laid on it, and on the glass plate the liquid F, to be experimented on. If K and F are connected with a wire D, which is held by means of an insulating handle and is of the same material as K, no result is observed on the removal of the wire, but the leaves separate immediately when the glass plate is raised.

It is thus shown that all metals brought into contact with liquids become electrified; that, further, a metal brought into contact with different liquids is sometimes electro-positive, sometimes electro-negative; and that the difference of potential possesses various values according to the nature of the liquid.

Metals show an entirely different behaviour towards liquids from what they do towards each other. Water, for example, is electro-positive towards zinc; if we suppose that water may take its place in Volta's electro-

motive series, then it must stand before zinc. It would then, however, follow that it must be far more strongly electro-positive towards copper, which comes after zinc in the series. Experience however teaches the contrary; water is electro-negative towards copper. Other liquids exhibit an entirely similar behaviour.

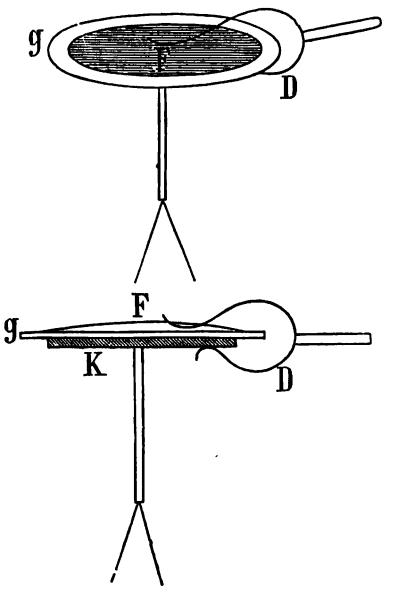

Fig. 69.

The liquids cannot therefore be placed in the electromotive series of Volta. In consequence of this property conductors are divided into two classes—into conductors of the first class and into conductors of the second class. The first class obeys the law of the electromotive series, and includes chiefly the metals; the second does not follow the law of the electromotive series, and includes the liquids.

If a liquid (Fig. 70) stands on one side in contact with a zinc plate, on the other side with a copper plate—such a connection of conductors is named the galvanic element—then electrical separating forces appear at the surfaces of contact, whereby the conductors become raised to definite potentials. If we name the potentials V_z, V_f, and V_k, we obtain the equations

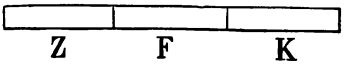

Fig. 70.

$$V_z - V_f = Z|F, \quad V_f - V_k = F|K,$$

therefore by addition

$$V_z - V_k = Z|F + F|K.$$

$Z|F$ and $F|K$ are the electromotive forces between the zinc and liquid and between the liquid and copper respectively. If we name the difference of potential $V_z - V_k$, which exists between the ends of the element, *the electromotive force of the element* Z, F, K, we can express the last equation in the following manner: *The electromotive force of the element* Z, F, K *is equal to the sum of the electromotive forces existing between the separate conductors.*

If we connect one end of the element with the earth, the zinc end for example, then $V_z = 0$, and, therefore,

$$-V_k = Z|F + F|K.$$

The difference of potentials of the ends of the chain remains the same as before, because $Z|F$ and $F|K$ are unchangeable electromotive forces; on the other hand, the value of the potential on the copper has become changed.

SECTION 46.—**The Galvanic Current and Combinations of Elements.**

If we place two plates of metal—for example, a zinc plate and a copper plate—together, each will be raised

to a definite potential, so that a difference of potential occurs between them.

If the potential on the zinc plate is V_z, and that on the copper plate V_k, then the equation

$$V_z - V_k = Z|K$$

follows. An electrical force, however, acts between two points which are at different potentials, the magnitude of which force, when the points lie very near one another, is given by the quotient of the potential-difference by the distance of the points. When we connect the two points by a conductor, an electric current takes place which conveys positive electricity from places of lower potential to places of higher potential, and causes the negative to move in the contrary direction, and which lasts as long as a difference of potential exists. Now one might think that if the zinc plate and the copper plate were connected by a metallic wire, *e.g.* by a copper wire, that a current of the kind designated would make its appearance. But this is not the case. If, for instance, we suppose the copper wire D (Fig. 71) connected with the zinc plate, an electrical separating force arises between it and this plate at the place of contact, which force is equal and opposite to that between the zinc and copper. We have here the case considered previously, viz. a series of conductors of the first class which begins and ends with the same metal. There exists, therefore, no difference of potential between the wire D and the plate of copper K. If we connect D and K by another copper wire, there results, since no new separating forces appear, *not the slightest current.* In like manner no current takes place if I connect the two plates by a zinc wire.

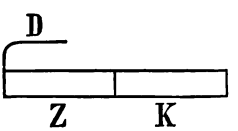

Fig. 71.

If I connect the two plates by a wire of another material, *e.g.* by a silver wire, then the effect is just the same as if I had connected the zinc plate with a silver wire and the copper plate with a silver wire, and had laid both wires on each other. Now since these two wires form the beginning and the end of a series of conductors of the first class, and are of the same material, they will acquire the same potential, and for this reason will not give rise to a current. Thus, when we connect the zinc and copper plates with a conductor of the first class, no current can take place.

The same thing holds good when we, in place of zinc and copper, choose other pairs of metals, and accordingly also when we join an entire series of metals to form a circuit, because the potential-difference between the first and the last metal is exactly the same as if we had brought these metals into direct contact with one another.

We thus arrive at the following proposition: *No electric current can circulate in a closed circuit of conductors of the first class.*

The state of matters is entirely different if we form a circuit of conductors of class I and class II. The conductors of the second class no longer obey the law of the electromotive series; the former relations can, therefore, no longer occur. Let us first consider a junction of zinc Z, liquid F, and copper K (Fig. 72), and imagine a copper wire D connected with the zinc Z. Let the values of potential on K, F, Z and D be denoted by V_k, V_f, V_z, and V_d, then

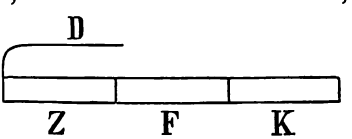

Fig. 72.

$$V_f - V_k = F|K, \quad V_z - V_f = Z|F, \quad V_d - V_z = K|Z,$$

or, by addition,

$$V_d - V_k = K|Z + Z|F + F|K.$$

Now since the liquids or the conductors of class II do not obey the electromotive law, $Z|F+F|K$ will not be equal to $Z|K$; consequently the sum

$$K|Z+Z|F+F|K, \quad \text{or} \quad -Z|K+Z|F+F|K$$

will be different from zero, and the wire D and the copper K possess a difference of potential whose magnitude is given by the sum of the electromotive forces of the circuit. If we connect D and K by a copper wire, electricity flows through this, the positive from positions of lower potential to positions of higher potential, the negative in the contrary direction.

The ends of an element are named its poles. If the copper plate K is connected with a copper wire D′, this will have the same potential as K. The wires D and D′ then form the ends or poles of the element; the former is called, on account of its connection with the zinc plate, the zinc pole, the latter, on account of its connection with the copper plate, the copper pole.

If F is dilute sulphuric acid, then positive electricity flows over from K through D′ and D to Z, from Z through F to K; the negative on the other hand from Z through D and D′ to K, from K through F to Z. These quantities are strictly equal in amount. Their passage through the circuit in opposite directions forms that condition which we name a current. The direction of the current is always determined by the direction in which the positive electricity flows. In the case considered the current goes outside the liquid from the copper pole to the zinc pole, and it may help to keep this in remembrance if we observe that the letter K occurs earlier in the alphabet than the letter Z.

We now consider three elements: the first may consist of the metals A and B and the liquid F, the second of the metals A and C and the liquid F, and the third

of the metals B and C and the liquid F (Fig. 73). The liquid F is the same in all the three elements. D_b and D_c denote wires of the same material as the plates B and C respectively. Let the potentials on D_b and B in the first element, D_c and C in the second element, and D_c and C in the third element be denoted respectively by V_{db}, V_b; V_{dc}, V_c; $V_{dc'}, V_{c'}$.

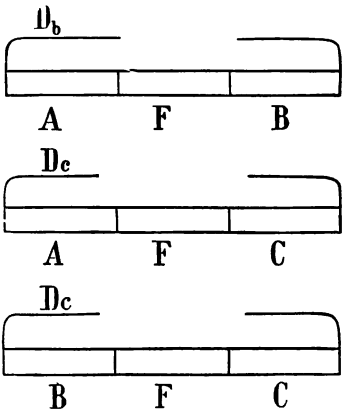

Fig. 73.

From the foregoing we find for the electromotive force of the first element the equation

$$V_{db} - V_b = B|A + A|F + F|B,$$

for the second element

$$V_{dc} - V_c = C|A + A|F + F|C,$$

and for the third element

$$V_{dc'} - V_{c'} = C|B + B|F + F|C.$$

If we subtract the first equation from the second there results

$$(V_{dc} - V_c) - (V_{db} - V_b) = C|A - B|A + F|C - F|B.$$

Now $-B|A = A|B$, and $-F|B = B|F$.

We therefore obtain

$$(V_{dc} - V_c) - (V_{db} - V_b) = C|A + A|B + B|F + F|C,$$

or, because the conductors A, B, C, as being conductors of the first class, obey the electromotive law, and in consequence the relation

$$C|A + A|B = C|B$$

exists, therefore

$$(V_{dc} - V_c) - (V_{db} - V_b) = C|B + B|F + F\,C,$$

or

$$(V_{dc} - V_c) - (V_{db} - V_b) = (V_{dc'} - V_{c'}),$$

i.e. in words, the electromotive force of the second element diminished by that of the first element is equal to the electromotive force of the third element, or, *if we connect a metal* A *with two other metals* B *and* C *and a liquid* F *to form two elements, then the electromotive force of the second element diminished by that of the first is equal to the electromotive force of an element which is composed of the two last metals* B *and* C *and the liquid* F.

If we have the elements zinc + sulphuric acid + platinum, and zinc + sulphuric acid + silver, then the difference of the electromotive forces of the two is equal to the electromotive force of the element, platinum + sulphuric acid + silver.

If three elements, as in Fig. 74, are connected to-

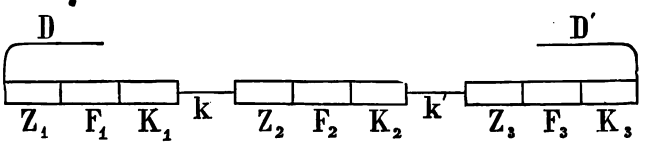

Fig. 74.

gether in such a manner that the copper plate of the first is connected by a copper wire *k* with the zinc plate of the second, the copper plate of the second by a copper wire *k'* with the zinc plate of the third, and the liquid in all the three circuits is the same, then the elements are said to be connected in series. How are the potentials related in such a case? We attach to the zinc plate of the first element a copper wire D, and we call the potential on D V_d, the potential on the zinc plates V_{1z}, V_{2z}, V_{3z}, on the copper plates V_{1k}, V_{2k}, V_{3k}, and in the liquid V_{1f}, V_{2f}, V_{3f}. The wire D forms the zinc pole, the copper wire D', which, on account of its connection with K_3, has the potential V_{3k}, forms the copper pole of the battery. The wire *k* has the same

potential as K_1, and the wire k' the same potential as K_2. We therefore obtain the following equations:

$$V_d - V_{1z} = K|Z, \quad V_{1z} - V_{1f} = Z|F, \quad V_{1f} - V_{1k} = F|K,$$
$$V_{1k} - V_{2z} = K|Z, \quad V_{2z} - V_{2f} = Z|F, \quad V_{2f} - V_{2k} = F|K,$$
$$V_{2k} - V_{3z} = K|Z, \quad V_{3z} - V_{3f} = Z|F, \quad V_{3f} - V_{3k} = F|K.$$

If we add the equations of the first row by taking all the expressions standing on the left and all those standing on the right, there results

$$V_d - V_{1k} = K|Z + Z|F + F|K.$$

We likewise obtain from the second row

$$V_{1k} - V_{2k} = K|Z + Z|F + F|K,$$

and from the third row

$$V_{2k} - V_{3k} = K|Z + Z|F + F|K.$$

If we now add these three equations we obtain for the difference of potentials of the poles of the battery the equation

$$V_d - V_{3k} = 3(K|Z + Z|F + F|K).$$

The electromotive force of the battery is therefore three times as great as that of a single element.

The method of investigation, which we have here employed for three elements, is equally applicable when we have to deal with any number of elements which are connected in series. If the number of elements is n, the electromotive force of the battery is n times as great as that of a single element.

If we connect one pole with the earth this cannot cause any alteration as regards the difference of potentials between the two poles, because this is determined by the unalterable forces causing separation in the battery. When therefore one pole is at the potential zero the potential of the other pole is numerically equal to the difference of potentials.

We will now connect three elements with one another, so that all the zinc plates are connected together by a zinc wire and all the copper plates by a copper wire; the elements are then said to be *connected abreast* (Fig. 75). To one of the zinc plates the copper wire D is attached, to one of the copper plates the copper wire D′; the former forms the zinc pole, the latter the copper pole. Since the zinc plates are all connected so as to form one conductor, and at the surfaces of contact of the wires no separating forces act, the same potential must exist on all. There is the same liquid in all the three elements; hence the same separating forces, and therefore the same differences of potential, exist between the liquid and the metals, or the potential of the three liquids is the same.

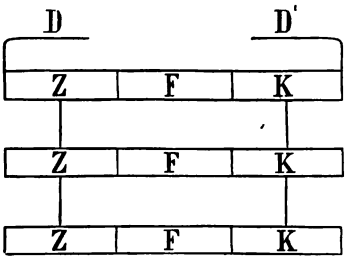

Fig. 75.

If we put V_d for the potential of the wire D, V_z for that of the zinc plate, V_k for that of the copper plate, and V_f for that of the liquid, we obtain the equations

$$V_d - V_z = K|Z, \quad V_z - V_f = Z|F, \quad V_f - V_k = F|K,$$

whence the equation

$$V_d - V_k = K|Z + Z|F + F|K,$$

i.e. the difference of potential of the poles, or *the electromotive force of the battery, which consists of three elements connected abreast, is equal to the electromotive force of a single element.**

* If the zinc plates are connected together by copper wires the new separating forces introduced by this arrangement cause no change of condition, because these are incapable of producing any difference of potential between the zinc plates.

The same naturally holds good if we connect any number of elements abreast.

If we connect 2, 3, 4, . . . n *elements in series, the electromotive force of the battery will be* 2, 3, 4, . . . n *times as great; if on the other hand we connect them abreast, the electromotive force always remains the same, viz. that of a single element.*

To the elements which we have here mentioned belong, amongst others, Volta's (copper, zinc, acidulated water), Smee's (amalgamated zinc,* platinised silver, *i.e.* silver covered over with spongy platinum, dilute sulphuric acid), and Bunsen's (zinc, carbon, solution of potassium dichromate). The solution of this element is prepared as follows: to 6.047 litres of water is added

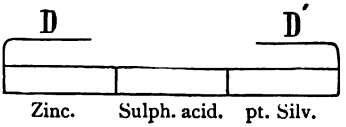

Fig. 76.

0.6282 litre of hydrogen sulphate and 618.2 grammes of powdered potassium dichromate.

Let us now consider Smee's element (Fig. 76). The copper wires D and D′ form the poles of the circuit; let their potentials be denoted by V_d and $V_{d'}$. Their difference of potential, or the electromotive force of the circuit, is given by the equation

$$V_d - V_{d'} = \text{copper}|\text{zinc} + \text{zinc}|\text{sulphuric acid} + \text{sulphuric acid}|\text{platinum} + \text{platinum}|\text{silver} + \text{silver}|\text{copper},$$

* Amalgamated zinc is always used because amalgamated zinc is not dissolved by dilute sulphuric acid—the zinc is merely consumed in consequence of the electrolytic process. Besides, amalgamated zinc is electropositive towards ordinary zinc.

or, because in accordance with the electromotive law,

$$\text{platinum}|\text{silver} + \text{silver}|\text{copper} + \text{copper}|\text{zinc} = \text{platinum}|\text{zinc},$$

by the equation

$$V_d - V_{d'} = \text{platinum}|\text{zinc} + \text{zinc}|\text{sulphuric acid} + \text{sulphuric acid}|\text{platinum}.$$

As we see, the electromotive force of the element is determined solely and only by the sum of the electromotive forces between the conductors composing it—platinum, zinc, and sulphuric acid.

Exactly the same holds good in regard to Bunsen's element. In this the electromotive force is given by the sum

$$\text{carbon}|\text{zinc} + \text{zinc}|\text{solution} + \text{solution}|\text{carbon}.$$

SECTION 47.—Galvanic Elements with two Liquids.

Just as two metals, and one metal and a liquid, give rise to an excitation of electricity, so also do two liquids by mutual contact, one liquid becoming charged with positive and the other with negative electricity. If a circuit is formed with two liquids instead of with one, the following relations hold good:

If (Fig. 77) Z is a zinc plate, K a copper plate, if further D and D′ are copper wires, and F and *f* two different liquids, then we obtain, when we designate the potentials on D, Z, F, *f* and K by V_d, V_z, V_f, $V_{f'}$ and V_k, the equations

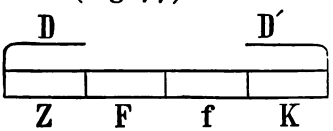

Fig. 77.

$$V_d - V_z = K|Z, \qquad V_z - V_f = Z|F,$$
$$V_f - V_{f'} = F|f, \qquad V_{f'} - V_k = f|K.$$

If we add these, there results

$$V_d - V_k = K|Z + Z|F + F|f + f|K.$$

Since the copper wire D′ possesses the same potential as K, $V_d - V_k$ is the diference of potential of the poles D and D′, or the electromotive force of the element. This is, as one sees, equal to the sum of the electromotive force between the two metals, of the electromotive force between the two liquids, and the electromotive forces between the metals on the one hand and the liquids on the other.

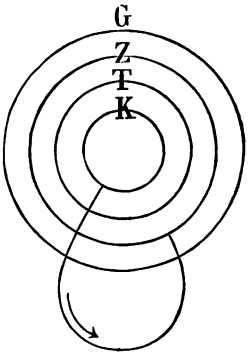

Fig. 78.

If we connect n such elements together in succession, the difference of potential of the poles, or the electromotive force of the battery, will be as before n times as great as that of a single element; if on the other hand we connect them abreast, the electromotive force remains equal to that of a single element.

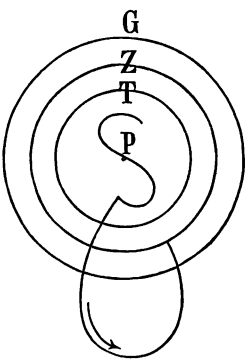

Fig. 79.

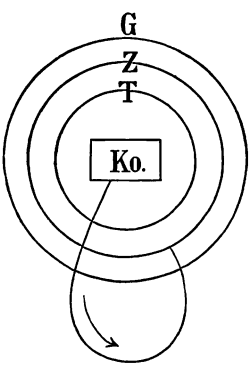

Fig. 80.

Daniell's element (Fig. 78) is of the kind described. G denotes a glass cylinder, Z a zinc cylinder, T a porous earthenware cell, and K a copper cylinder. The

liquid in the earthenware cell, *i.e.* the liquid *f*, is a solution of copper vitriol, the liquid outside the earthenware cell or the liquid F is dilute sulphuric acid. If in this element we replace the copper by an S-shaped platinum plate, and the copper vitriol solution by concentrated nitric acid, we obtain Grove's element (Fig. 79). And if we substitute in this carbon for platinum, then Bunsen's element is obtained (Fig. 80).

The current through the wire in Daniell's element goes from the copper to the zinc, in Grove's element from the platinum to the zinc, and in Bunsen's element from the carbon to the zinc.

SECTION 48.—**Ohm's Law.**

We will now consider the conditions of a closed circuit, *e.g.* a Daniell circuit. Let Fig. 81 represent the plan of its arrangement. Let D, D′, the poles of the circuit, be copper wires, which are in contact with the zinc and copper plates at the surfaces A and B respectively. If the potential of the wire D be V_d and that of the wire D′ be $V_{d'}$, then the difference $V_d - V_{d'}$ represents the electromotive force of the circuit. If we connect the wires D and D′ by a copper wire *d*, naturally no separating force occurs at the surfaces of contact; instead of this, however, because two places of different potentials are connected together, electrical forces will come into play, which are the greater the greater the difference of potentials, and which cause the positive electricity to move from positions of lower potential to positions of higher potential, but the negative electricity

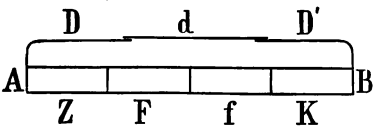

Fig. 81.

to move in an opposite direction, and thus, in other words, call forth an electric current. This current tends to diminish the difference of potentials, and it would make this equal to zero, *i.e.* would equalise the potentials of both poles, if there were no further forces present which tend to restore it.

The several portions of the circuit are, as before, in mutual contact; hence separating forces arise that move the positive electricity in one and the negative in the opposite direction. These separated electricities endeavour not only to become again united, but also directly oppose a further separation. If the poles D and D′ are not connected—that is, if the circuit is open, the electrical forces of the separated electricities will more and more diminish the effect of the separating forces and finally neutralise them. The latter condition will be reached when the poles have acquired such a difference of potentials that this is equal to the sum of the electromotive forces of the circuit, or that the equation

$$V_d - V_{d'} = K|Z + Z|F + F|f + f|K$$

exists. The condition of the circuit is then a condition of equilibrium.

If the poles D and D′ are connected together, then the electric current will tend to diminish the difference of potentials mentioned, and therefore will tend to disturb the condition of equilibrium. The least disturbance immediately evokes a fresh action on the part of the separating forces, which endeavours to maintain the difference of potentials at the surfaces of contact between D and Z, Z and F, F and *f*, *f* and K, and by this to bring the difference of potentials of the poles to its original height. Since the current goes on continuously the action of the separating forces also goes on continuously, and since the former continuously causes

combination of the opposite electricities in the external conducting wire, the latter will continuously separate fresh quantities of electricity. In this manner, therefore, a continuous current occurs, not only in the connecting wire *d*, but also in the element itself, everywhere in the same direction.

The conditions under which the current circulates are at every instant the same; accordingly a so-called *stationary condition* comes about, by reason of which the conditions in the several parts of the circuit do not now change with time, but the potential of the electricity at every point of the circuit has a definite value, which however changes from point to point.

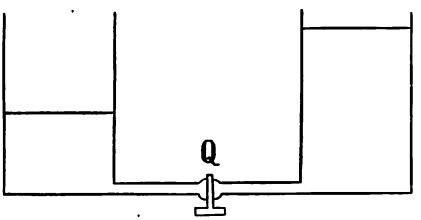

Fig. 82.

This condition can in some degree be illustrated by the following analogous case: let two vessels filled to different heights with water be connected together by a tube (Fig. 82). When the stopcock Q is opened, water flows from the vessel in which the height of the water is greater into the other vessel, and continues to flow until both levels have been equalised. The conditions are naturally different if we represent both vessels to be infinitely wide and the tube to be very narrow. The levels are not then changed by the current, the difference of levels or the condition of the flow remains always the same.

We have seen that the potential at all points of the

circuit has a value independent of time, which value changes from point to point; the question then arises, What is the law of this change? If we pass from A along D, *d*, D′ to B, we nowhere find on this space surfaces of contact in which separating forces are present; we can consequently assume that the potential from A to B changes continuously, *i.e.* it increases or diminishes by an infinitely small amount if we advance an infinitely small distance. The different parts of the connecting wire AB possess different potentials. If we now suppose the corresponding equipotential surfaces constructed, these cut the wire in certain surfaces. If the cross-sections of the connecting wire are everywhere very small, we can without further consideration look upon the equipotential surfaces as coinciding with the cross-sections. Between two such cross-sections there is a definite piece of wire enclosed. Let us now consider three such pieces of wire contiguous to each other, *ab*, *bc*, *cd*, and let their lengths be very small (Fig. 83).

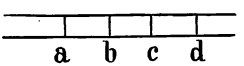

Fig. 83.

We will designate the values of the potential at the cross-sections *a*, *b*, *c* and *d* respectively by V_a, V_b, V_c, V_d. Between the cross-sections *a* and *b* an electrical force acts, which is equal to the potential-difference divided by the distance, and therefore has the value

$$\frac{V_a - V_b}{ab}.$$

Between the cross-sections *b* and *c* an electrical force acts of the magnitude

$$\frac{V_b - V_c}{bc},$$

and between *c* and *d* an electrical force of the magnitude

$$\frac{V_c - V_d}{cd}.$$

These forces move the two electricities in different directions, the positive in the one and the negative in the other. Let us assume that the positive electricity is driven towards the right and the negative towards the left, and let us first consider the part of the conductor *bc*. From the cross-section *c* positive electricity passes out, from *b* negative, and in equal quantity.

These quantities depend evidently on the magnitude of the cross-section; if this becomes 2, 3, . . . *n* times as great, 2, 3, . . . *n* times as great a quantity of electricity will be separated, provided that at all points of the cross-section the same force acts on the unit of positive electricity. If the cross-section has unit area equal to one cubic centimetre, we will name the quantity of positive electricity which passes through the cross-section in the one direction in a second, diminished by the amount of negative electricity which passes in the same time through the cross-section in an opposite direction, *i.e.* the total quantity of electricity which passes through the unit cross-section in a second, the *current* or *the specific-current intensity*. Since the quantity of positive electricity which passes through the cross-section is equal and opposite to the negative, the sum of the two equals zero, and their difference is equal to double the quantity of positive electricity.

The current or the specific-current intensity evidently becomes 2, 3, . . . *n* times as great when the force acting on the unit of positive electricity becomes 2, 3, . . . *n* times as great—it is therefore proportional to this force; we will for the portion *bc* of the conductor put it equal to

$$k \cdot \frac{V_b - V_c}{bc}.$$

The constant factor *k* depends only on the material

of the conductor, and is called the conductivity. Its reciprocal value, *i.e.* $1/k$, is termed the specific resistance of the conductor; we will denote it by r. If the cross-section of the part *bc* of the conductor possesses the magnitude ω, and if the force which acts on the unit of positive electricity has at all points the same value and the same direction, then the total quantity of electricity which passes through the cross-section per second is equal to

$$k\omega \frac{V_b - V_c}{bc}.$$

This quantity is termed the *current intensity* or the *current strength* at this place. If we term the quotient $\frac{V_b - V_c}{bc}$, the rate of fall of the potential in the portion *bc* of the conductor, we can say: *The strength of the current at any part of the circuit is equal to the product of three factors, viz. that of the conductivity, the cross-section, and the rate of fall of potential.*

Just as the strength of the flow of water or gas through pipes is not measured by the velocity of the water or the gas, but by the quantity which flows out in a second, so here also the strength of the electric current is measured by the quantity of electricity which passes through the cross-section per second.

From the part *bc* of the conductor there accordingly issue equal and opposite quantities of electricity. Precisely the same occurs in regard to the neighbouring parts *ab* and *cd* of the conductor. From *ab* the positive electricity issues towards the right, and consequently passes into *bc*, whilst negative electricity from the portion *cd* of the conductor going towards the left enters *bc*. If now the positive electricity from *ab* were greater or less than this negative electricity, the element

bc of the conductor would experience a positive or a negative charge, which, if the current continued circulating in this manner, must become constantly greater and greater. Then, however, the potential in *bc* would alter with time, which is contrary to the supposition, because we have assumed that the condition being everywhere a stationary one, has become independent of time. It follows therefore that in the portions *ab* and *cd* of the conductor equal quantities of electricity are separated, or that in both equal quantities of electricity pass through each cross-section in every second, or that in both the same strength of current prevails. We can apply precisely similar reasoning to all parts of the conducting circuit, viz. not only to the connecting wire, but also to the separate parts of the element itself; we therefore come to the conclusion that *in all parts of the conducting circuit the strength of the current is the same.*

Now we have seen that the strength of the current at any one position is determined by the product of the conductivity, the cross-section, and the rate of fall of potential at that position; this product must, therefore, be the same everywhere in the circuit, or the *rates of fall of potential at the various parts of a circuit must be to each other inversely as the products of the conductivity and the cross-section of the parts.*

If the wire between A and B has everywhere the same cross-section ω, then the rate of fall of potential is the same throughout, because the conductivity is everywhere the same, or the potential changes from point to point along the wire in such a manner that if we advance along the wire by equal spaces the potential changes by the same amount, therefore, just as in the case of uniform motion distance changes with time; in other words, it *follows the law of a straight line.* In

Fig. 84 AB represents the wire. At every cross-section the corresponding value of potential is applied as an ordinate. The terminal points of the ordinates give, when joined together, the straight line CD, which represents the course of the potential. Its slope or its rate of fall towards AB depends on the conductivity k and the cross-section ω. If we name the rate of fall G_d and the length of the wire l_d, then

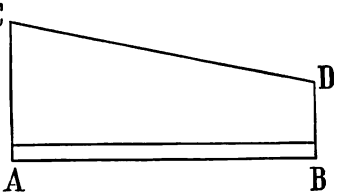

Fig. 84.

$$AC + G_d l_d = BD.$$

Here the rate of fall G_d, if we go from A to B, is to be taken negative. From the last equation we obtain

$$G_d = \frac{BD - AC}{l_d},$$

or, *the rate of fall of potential along a portion of a conductor of uniform cross-section is directly proportional to the difference of potentials of its ends, but inversely proportional to the length of the conductor.*

For the strength of the current we found the expression $k_d \omega_d G_d$, when we denote by k_d the conductivity and by ω_d the cross-section of the conductor AB. According to what has been stated, the strength of the current is therefore equal to

$$k_d \omega_d \frac{BD - AC}{l_d},$$

or

$$\frac{BD - AC}{\frac{l_d}{k_d \omega_d}}.$$

The quantity $\frac{l_d}{k_d \omega_d}$, or, if we put $\frac{1}{k_d} = r_d$, $r_d \frac{l_d}{\omega_d}$ is called

the resistance of the conductor. This is, as we have seen, directly proportional to the length l_d and to the specific resistance r_d, inversely proportional to the cross-section. If we denote this resistance by R_d, we get for the strength of the current the expression

$$\frac{BD - AC}{R_d},$$

and in consequence the following proposition: *The strength of the current in the conductor* AB *is directly proportional to the difference of the values of the potential prevailing at the ends, but inversely proportional to the resistance of the conductor.*

If $l_d = 1$, and $BD - AC = 1$, therefore $G_d = 1$; and if further $\omega_d = 1$, then the strength of the current is equal to k_d, or *the conductivity is equal to that strength of current which occurs in the conductor at a cross-section of unit area, when the rate of fall of potential is unity.*

At any other part of the circuit, for example the zinc, the strength of the current, in order that the condition of the motion may remain stationary, must possess the same value. If the zinc has everywhere the same cross-section, then the potential here also changes according to the law of a straight line, only its rate of fall will in general be different from that in the external conducting wire.

Let us still consider the surfaces of contact of two parts of the circuit. At the place of contact of the wire D with the zinc Z (Fig. 85) we have evidently two cross-sections; the one belongs to the wire, the other to the zinc. Let the magnitude of the former be denoted by ω_d, that of the latter by ω_z. In order that the state of things may remain stationary, the strength of the current must be the same in both cross-sections. The

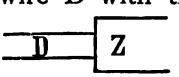

Fig. 85.

strength of the current is, as we have seen, equal to the product of the conductivity of the cross-section and the force which there acts on the unit of positive electricity. At the surface of contact there now acts a separating force—we will designate it by S—which seeks to move the positive electricity in the one direction, the negative in the opposite. This separating force will, however, be neutralised by the action of the potential-difference which exists at the same place, and which possesses the same magnitude as if the circuit were open and no current circulated. The current therefore at the cross-section ω_d is only determined by the rate of fall of the potential in the wire D, and likewise the current in the cross-section ω_z is only determined by the rate of fall of the potential in the zinc Z; and in accordance with this we obtain for the strength of the current in ω_d the expression

$$k_d \omega_d G_d,$$

and for the strength of the current in ω_z, if we denote the rate of fall of potential in the zinc (which like the conductor D may be supposed to possess a uniform cross-section) by G_z and the conductivity by k_z, we obtain the expression $k_z \omega_z G_z$.

Both expressions must be equal to one another, therefore the equation

$$k_d \omega_d G_d = k_z \omega_z G_z$$

must exist, or there is the proportion

$$G_d : G_z = k_z \omega_z : k_d \omega_d,$$

i.e. *the rates of fall of potential in two different conductors, each of which possesses a constant cross-section, are to one another inversely proportional to the products of the corresponding cross-sections and conductivities.*

From the last equation the value for G_z is obtained,

$$G_z = \frac{k_d \omega_d G_d}{k_z \omega_z} \qquad (a).$$

To this equation a second is to be added. If we denote the potential in ω_d by V_d, and in ω_z by V_z, then

$$V_d - V_z = K|Z \qquad (b),$$

or

$$V_z = V_d - K|Z.$$

If the potentials in D and Z follow the law of a straight line, and if its course in D is known, we can find the course of the potential in Z by means of the equations (*a*) and (*b*), since the equation (*a*) determines the rate of fall, and equation (*b*) the initial ordinate (at the surface of contact between D and Z).

If l_z is the length of the zinc conductor, the potential at the end of this is given by

$$V_z + G_z l_z.$$

At this end the zinc stands in contact with the liquid F. Exactly the same reasoning as before applies to this surface of contact. If F has everywhere the same cross-section, then the potential again follows the law of a straight line whose fall and initial ordinate (at the surface of contact between Z and F) are determined by two analogous equations.

If we assume that all portions of the circuit are so constituted *that each everywhere possesses the same cross-section, then the potential changes in every one according to the law of a straight line. In the passage from one to the other, however, it makes a leap, which is equal to the electromotive force acting at that place.*

In order to give a representation of the course of the potential in the entire circuit of the current, we imagine this to be cut across at the surface of contact of D and K, and stretched out straight; but at the same time we will imagine that the potentials are exactly the same as if the circuit were closed. In Fig. 86 we see these relations represented.

As a result of our discussion up to the present, we

have arrived at the proposition that the strength of the current is everywhere the same throughout the circuit, and also at the law according to which the potential changes in the separate divisions of the circuit. We will now determine the magnitude of the potential at any cross-sections of the circuit.

The electromotive forces of the circuit, namely, K|Z, Z|F, F|f,f|K, besides the lengths l_d, l_z, l_f, $l_{f'}$, l_k, the cross-sections ω_d, ω_z, ω_f, $\omega_{f'}$, ω_k, and the conductivities k_d, k_z, k_f, $k_{f'}$, k_d* of the parts of the circuit D, Z, F, f, K, are given. If we represent the rates of fall of potential in these portions of the circuit by G_d, G_z, G_f, $G_{f'}$, G_k, then, according to what has been stated above, the following equations hold good:

$$k_d\omega_d G_d = k_z\omega_z G_z = k_f\omega_f G_f = k_{f'}\omega_{f'} G_{f'} = k_d\omega_k G_k.$$

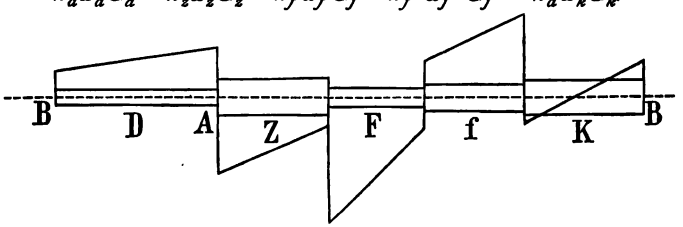

Fig. 86.

With the help of these equations we can determine all the remaining rates of fall of potential as soon as the rate of fall of potential in one part of the chain is known. We will denote the values of the potential at the beginning and end of the portions of the circuit relative to D by V_{1d} and V_{2d}, relative to Z by V_{1z} and V_{2z}, relative to F by V_{1f} and V_{2f}, relative to f by $V_{1f'}$ and $V_{2f'}$, and relative to K by V_{1k} and V_{2k}. We then obtain for the part of the circuit D

$$V_{1d} + l_d G_d = V_{2d} \qquad (1);$$

* The part of the conductor K is of the same material as D.

and at the surface of contact between D and Z

$$V_{2d} - V_{1z} = K|Z \qquad (2);$$

for the part Z of the circuit

$$V_{1z} + l_z G_z = V_{2z},$$

or

$$V_{1z} + l_z \frac{k_d \omega_d}{k_z \omega_z} G_d = V_{2z} \qquad (3);$$

and at the surface of contact between Z and F

$$V_{2z} - V_{1f} = Z|F \qquad (4);$$

for the part F of the circuit

$$V_{1f} + l_f G_f = V_{2f},$$

or

$$V_{1f} + l_f \frac{k_d \omega_d}{k_f \omega_f} G_d = V_{2f} \qquad (5);$$

and at the surface of contact between F and f

$$V_{2f} - V_{1f'} = F|f \qquad (6);$$

for the part f of the circuit

$$V_{1f'} + l_{f'} G_{f'} = V_{2f'},$$

or

$$V_{1f'} + l_{f'} \frac{k_d \omega_d}{k_{f'} \omega_{f'}} G_d = V_{2f'} \qquad (7);$$

and at the surface of contact between f and K

$$V_{2f'} - V_{1k} = f|K \qquad (8);$$

for the part K of the circuit

$$V_{1k} + l_k G_k = V_{2k},$$

or

$$V_{1k} + l_k \frac{k_d \omega_d}{k_d \omega_k} G_d = V_{2k} \qquad (9),$$

and at the surface of contact between K and D

$$V_{2k} - V_{1d} = 0 \qquad (10).$$

No separating force acts between K and D, because both are of the same material.

If we now add these ten equations, collecting together separately the expressions standing on the left and the expressions standing on the right, then the quantities V_{2d}, V_{2z}, V_{2f}, $V_{2f'}$, V_{2k} on the left and right, and the quantities V_{1d}, V_{1z}, V_{1f}, $V_{1f'}$, V_{1k} on the left will cancel each other, and we obtain

$$l_a G_d + l_z \frac{k_d \omega_d}{k_z \omega_z} G_d + l_f \frac{k_a \omega_d}{k_f \omega_f} G_d + l_{f'} \frac{k_d \omega_d}{k_{f'} \omega_{f'}} G_d$$
$$+ l_k \frac{k_d \omega_d}{k_d \omega_k} G_d = K|Z + Z|F + F|f + f|K.$$

If we write the term $l_d G_d$ in the form

$$l_d \frac{k_d \omega_d}{k_d \omega_d} G_d,$$

and take out $k_d \omega_d G_d$ from all the terms standing on the left, the result is

$$k_d \omega_d G_d \left(\frac{l_d}{k_d \omega_d} + \frac{l_z}{k_z \omega_z} + \frac{l_f}{k_f \omega_f} + \frac{l_{f'}}{k_{f'} \omega_{f'}} + \frac{l_k}{k_d \omega_k} \right)$$
$$= K|Z + Z|F + F|f + f|K.$$

The resistances of the parts of the circuit stand on the left in the bracket. If we denote these by R_d, R_z, R_f, $R_{f'}$, R_k, then

$$k_d \omega_d G_d = \frac{K|Z + Z|F + F|f + f|K}{R_d + R_z + R_f + R_{f'} + R_k} \qquad (11).$$

The strength of the current in the part D of the circuit, as we have seen, equals $k_d \omega_d G_d$, and the strength of the current at every cross-section of the circuit possesses this value. If we put it equal to I we obtain the final equation,

$$I = \frac{K|Z + Z|F + F|f + f|K}{R_d + R_z + R_f + R_{f'} + R_k} \qquad (12).$$

By this Ohm's principal law is expressed. On the right, in the numerator, stands the sum of the separate electromotive forces in the circuit, *i.e.* the resultant electromotive force of the circuit; in the denominator the sum of the resistances of the separate parts of the circuit, *i.e.* the total resistance of the circuit. *The strength of the current is therefore equal to the quotient of the electromotive force by the total resistance of the circuit.*

By virtue of equation (11) we obtain by summation

of the first nine equations equation (10); this equation is therefore not independent of the others, and there remain consequently only nine equations for the determination of the ten values of potential, V_{1d}, V_{2d}, V_{1z}, V_{2z}, V_{1f}, V_{2f}, $V_{1f'}$, $V_{2f'}$, V_{1k}, V_{2k}. With the help of these nine equations we can express nine values of potential in terms of the tenth, *e.g.* by V_{1d}; this, however, is not further determined and is completely arbitrary. We obtain the following equations:

$$V_{2d} = V_{1d} + R_d I$$
$$V_{1z} = V_{1d} + R_d I - K|Z$$
$$V_{2z} = V_{1d} + (R_d + R_z) I - K|Z$$
$$V_{1f} = V_{1d} + (R_d + R_z) I - K|Z - Z|F$$
$$V_{2f} = V_{1d} + (R_d + R_z + R_f) I - K|Z - Z|F$$
$$V_{1f'} = V_{1d} + (R_d + R_z + R_f) I - K|Z - Z|F - F|f$$
$$V_{2f'} = V_{1d} + (R_d + R_z + R_f + R_{f'}) I - K|Z - Z|F - F|f$$
$$V_{1k} = V_{1d} + (R_d + R_z + R_f + R_{f'}) I - K|Z - Z|F - F|f - f|K$$
$$V_{2k} = V_{1d}.$$

It follows from this that if V_{1d} changes by a definite amount, all the other values of potential experience exactly the same change. The value of one of the nine potential values will, as we have seen, be found, if we add to the potential V_{1d} the product of the strength of the current I, and the resistance of that part of the circuit which is situated between the cross-section of the potential V_{1d} and the cross-section under consideration, and subtract from this the sum of the intervening electromotive forces.

If we consider a cross-section which does not coincide with a surface of contact of two conductors, *e.g.* a cross-section on the part F of the circuit, which is distant x from the place to which V_{1f} belongs, then the

potential at that position—we will call it V_x—is given by $$V_x = V_{1f} + G_{f.x},$$

or $$V_x = V_{1f} + \frac{x}{k_f \omega_f} I.$$

Now $\frac{x}{k_f \omega_f}$ is nothing else than the resistance of the portion x of the conductor F. If we name this resistance R_x then $V_x = V_{1f} + R_x I$,

or, by virtue of the above,

$$V_x = V_{1d} + (R_d + R_z + R_x)I - K|Z - Z|F.$$

The proposition which we have already stated in regard to the nine values of potential is therefore of general application. It runs as follows: *The value of the potential at any place whatever of the circuit will be found if we add to the value of the potential* V_{1d} *the product of the strength* I *of the current and of the resistance of that part of the circuit which is situated between the cross-section under consideration and that having the potential* V_{1d}, *and subtract from this the sum of the electromotive forces found in this part of the circuit.*

This law is true not only for a circuit of two metals and two liquids, but is of universal application for a galvanic circuit composed of any number of parts.

The resistance R_d of the external connecting wire is named the *external resistance;* on the other hand the resistance $R_z + R_f + R_{f'} + R_k$, *i.e.* the resistance in the element, is named the *internal.*

We will now apply the result obtained in equation (12) to the combinations of elements considered in section 46 in series and abreast.

If there are n equal elements of the kind last considered connected in series, the electromotive force of the battery is n times as great as that of a single element, therefore equal to $$n(K|Z + Z|F + F|f + f|K).$$

If we make use of the same connecting wire as with a single element, then the external resistance is the same, viz. R_d. The internal resistance has, however, become greater, viz. n times as great as that of a single element, because the resistances are directly proportional to the lengths. The strength of the current is now given by the equation

$$I = \frac{n(K|Z + Z|F + F|f + f|K)}{R_d + n(R_z + R_f + R_{f'} + R_k)}.$$

If R_d is very small we gain nothing by this mode of connecting elements, but advantage ensues when R_d is very great, so that as compared with it the internal resistance vanishes. *In this case the strength of the current is raised nearly* n-*fold.*

If we have n equal elements connected abreast, then the electromotive force of the battery is equal to that of a single element, or

$$K|Z + Z|F + F|f + f|K.$$

If we again use the same connecting wire, the external resistance then remains the same, viz. R_d. The internal resistance has now, however, become smaller. Since the resistance of a given part of the circuit is inversely proportional to its cross-section, and the conductors of the same kind being connected together form one conductor of n-fold cross-section, the internal resistance of the battery will now be n times as small as that of a single element. We obtain therefore for the strength of the current the equation

$$I = \frac{K|Z + Z|F + F|f + f|K}{R_d + \frac{1}{n}(R_z + R_f + R_{f'} + R_k)}.$$

If the external resistance R_d is very great in comparison with the internal, we have gained by this mode

of connection almost nothing. *If, on the other hand,* R_d *is exceedingly small in comparison with the internal resistance, the strength of the current will then be almost* n *times as great.*

The question now immediately presents itself: How must a given number of elements be connected together in order that the strength of the current may acquire the greatest possible value? A simple calculation shows that the strength of the current acquires its greatest value when the *internal resistance of the battery is equal to the external resistance,* i.e. *equal to that of the connecting wire.* If we denote the internal resistance of an element by R, and if n is the number of the elements, then this condition will be satisfied if we always connect

$$\sqrt{n\frac{R}{R_d}}$$

elements abreast, and connect the groups so obtained in series.

SECTION 49.—**Continuation: Branching of the Current (Divided Currents).**

We have hitherto considered only the case in which the battery is closed by a single wire, and investigated the resulting conditions of the circuit. We will now discuss the case in which the connecting wire consists of several branches.

Let the connecting wire (Fig. 87) branch at the point A′ into three branches, 1, 2, 3, which unite again at the point B,′ and which are of the same material as the remaining part of the connecting wire, so that at the points A′ and B′ no separating forces make their appearance.

The separating forces in the circuit, which depend only on the internal constitution of the materials em-

ployed, are the ultimate cause of the current. If the materials remain the same, the separating forces also always remain the same; therefore also the conditions under which the current circulates. The consequence of this is that here also, just as previously, a stationary condition sets in, in consequence of which the relations in the separate parts of the circuit do not change with time. If we keep this firmly in view, we come, if we consider one of the branches, *e.g.* branch 1, exactly as before, to the conclusion that the strength of the current in all cross-sections of this branch must be the same. Now we saw that the strength of the current was found in a cross-section when we multiplied the rate of fall of potential prevailing there by the conductivity and the cross-section. The conductivity is the same for all branches. If we assume this likewise of the cross-section, then the rate of fall of potential, because the strength of the current has everywhere the same value, is everywhere the same, or the potential changes along the branch according to the law of a straight line.

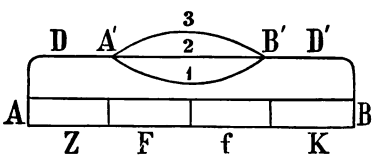

Fig. 87.

If l_1 is the length of branch 1, and if the potential at A' and B' has the values V_a and V_b, the rate of fall of potential in branch 1 equals

$$\frac{V_a - V_b}{l_1}.$$

If we denote the cross-section by ω and the conductivity by k, the strength of the current—we will name it i_1—is given by

$$k\omega_1 \frac{V_a - V_b}{l_1}.$$

Now $\frac{l_1}{k\omega_1}$ is nothing else than the resistance of branch 1. If we name this R_1, we obtain the equation

$$i_1 = \frac{V_a - V_b}{R_1}.$$

We can follow exactly the same method for branches 2 and 3. It is at the same time to be observed that the potential at the point A′ can have only one value, viz. the value V_a, and at the point B′ also only one value, viz. V_b. If, therefore, i_2 and i_3 denote the strengths of the currents, and R_2 and R_3 the resistances for branches 2 and 3, then

$$i_2 = \frac{V_a - V_b}{R_2}, \quad i_3 = \frac{V_a - V_b}{R_3},$$

or the proportion

$$i_1 : i_2 : i_3 = \frac{I}{R_1} : \frac{I}{R_2} : \frac{I}{R_3}$$

is true, *i.e. the strengths of the currents are to each other inversely as the corresponding resistances.*

If the strength of the current outside the branches is equal to I, then, because the current divides into three branch currents at the point A′, and these unite again at the point B′ to form one current of the strength I, the equation

$$I = i_1 + i_2 + i_3$$

follows.

If we suppose the branch system replaced by a single wire of the same material and of such a resistance R that outside A′B′ all remains the same, that therefore the strength of the current possesses the same magnitude and the potential everywhere the same value as before, then, because also at the points A′ and B′ the potential is the same, the strength of the current

in the new wire or the strength of the current I will be equal to

$$\frac{V_a - V_b}{R}.$$

If we substitute in the last equation but one the values for I, i_1, i_2, i_3 we obtain the relation

$$\frac{I}{R} = \frac{I}{R_1} + \frac{I}{R_2} + \frac{I}{R_3}.$$

The strength of the current is, as we have seen, equal to the electromotive force of the circuit divided by its total resistance. If we name the resistance of the circuit, with the exception of the new wire, U, and the electromotive force of the circuit E, then

$$I = \frac{E}{U + R}.$$

Since R is given by the foregoing equation, then the strength I of the current is determined by this. We can also determine the branch currents themselves. We have

$$I = \frac{V_a - V_b}{R},$$

and

$$i_1 = \frac{V_a - V_b}{R_1};$$

whence

$$\frac{I}{i_1} = \frac{R_1}{R}, \quad \text{or } i_1 = \frac{R}{R_1} \cdot I.$$

We find similarly

$$\frac{I}{i_2} = \frac{R_2}{R}, \quad \text{or } i_2 = \frac{R}{R_2} \cdot I,$$

and

$$\frac{I}{i_3} = \frac{R_3}{R}, \quad \text{or } i_3 = \frac{R}{R_3} \cdot I.$$

The laws which we have here ascertained for three branch currents are true in the same manner for any number of branch currents.

The laws in sections 48 and 49 were first unfolded by G. S. Ohm in his memorable book, *Die galvanische Kette mathematisch bearbeitet*, published at Berlin in 1827. He named potential "electroscopic force," and the difference of potential at the surface of contact of two conductors of different material "electrical tension," or "difference of the bodies."

Ohm's laws have been tested in the completest manner in all their parts by painstaking researches, and are acknowledged to be exact. Among the men who have tested them we may mention Ohm, Fechner, Pouillet, and Kohlrausch.

SECTION 50.—**Kirchhoff's Laws.**

With Ohm's laws are connected two laws which Kirchhoff stated in 1845, whereby the general problem of divided currents may be solved.

First law.—*If the wires* (Fig. 88) 1, 2, . . . n *meet together at a point, and if we designate the currents which flow towards the point as positive, and those which flow away from the point as negative, the sum of all these currents is equal to zero, or*

$$i_1 + i_2 + i_3 + \ldots i_n = 0.$$

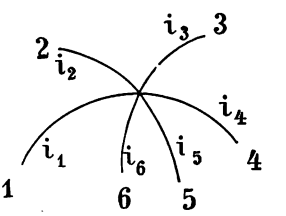

Fig. 88.

The proof is very simple. If there is a closed circuit, then a stationary condition takes place, in which there is no more change with time. Now, if the sum were not equal to zero, either more positive or negative electricity would flow to the meeting point than would flow away; positive or negative electricity would then collect at this point, and, because the conditions are constant, would increase continually in the course of time. But the potential

at the meeting point must then continually change in the course of time, which is contrary to the supposition.

Second law.—*If the wires* 1, 2, . . . n *form a closed figure, then the sum of the products of the strengths of the currents into the corresponding resistances is equal to the sum of all the electromotive forces which are found on the paths* 1, 2, . . . n, or, if we denote the resistances by $r_1, r_2 \ldots r_n$ and the electromotive forces by $C_1, C_2 \ldots C_n$, the equation

$$i_1 r_1 + i_2 r_2 + \ldots i_n r_n = C_1 + C_2 + \ldots C_n$$

follows.

We will go through the proof for three crossing points; the general proof is exactly similar. At the points *a*, *b*, *c* (Fig. 89) let separating forces act which produce the differences of potential C_a, C_b, C_c. Let l_1, l_2, l_3 be the lengths of the wires 1, 2, 3; r_1, r_2, r_3 their resistances; n_1, n_2, n_3 their rates of fall of potential; and i_1, i_2, i_3 the strengths of the currents.

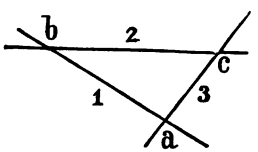

Fig. 89.

If the potential of wire 1 at the point *a* has the value m_1, the potential at the end of this wire or at the point *b* is equal to $m_1 + n_1 l_1$.

If the potential at the beginning of wire 2 is equal m_2, then

$$m_1 + n_1 l_1 - m_2 = C_b \qquad (1).$$

The potential at the end of wire 2, or at the point *c*, is $m_2 + n_2 l_2$. If the potential at the beginning of wire 3 has the value m_3, then

$$m_2 + n_2 l_2 - m_3 = C_c \qquad (2).$$

The potential at the end of wire 3 at the point *a* is equal to $m_3 + n_3 l_3$. Now the potential at the beginning of wire 1 equals m_1; we have, therefore,

$$m_3 + n_3 l_3 - m_1 = C_a \qquad (3).$$

If we add the equations (1), (2), and (3) we obtain

$$n_1 l_1 + n_2 l_2 + n_3 l_3 = C_a + C_b + C_c.$$

Now if the wires 1, 2, 3 have the cross-sections ω_1, ω_2, ω_3, and the conductivities are k_1, k_2, k_3, then

$$i_1 = k_1 \omega_1 n_1, \quad i_2 = k_2 \omega_2 n_2, \quad i_3 = k_3 \omega_3 n_3,$$

or, because

$$r_1 = \frac{l_1}{k_1 \omega_1}, \quad r_2 = \frac{l_2}{k_2 \omega_2}, \quad r_3 = \frac{l_3}{k_3 \omega_3},$$

$$i_1 r_1 = n_1 l_1, \quad i_2 r_2 = n_2 l_2, \quad i_3 r_3 = n_3 l_3.$$

If we substitute these values in the above equation, there results

$$i_1 r_1 + i_2 r_2 + i_3 r_3 = C_a + C_b + C_c,$$

and this is the second law.

We will apply the propositions to a case in which we have no separating forces at the crossing points. We will distinguish magnitudes which relate to the respective wires 1, 2, 3, 4, 5 (Fig. 90) by the corre-

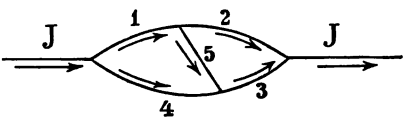

Fig. 90.

sponding numbers used as suffixes. The arrows show the directions of the currents. According to the first law we obtain

$$I - i_1 - i_4 = 0, \quad i_2 + i_3 - I = 0,$$

$$i_1 - i_2 - i_5 = 0, \quad i_4 + i_5 - i_3 = 0.$$

According to the second law

$$i_1 r_1 + i_5 r_5 - i_4 r_4 = 0,$$

$$i_2 r_2 - i_3 r_3 - i_5 r_5 = 0.$$

If we now assume that no current flows in the cross wire 5 (bridge), then $i_5 = 0$; hence

$$i_1 = i_2 \text{ and } i_3 = i_4,$$

and $$i_1 r_1 = i_4 r_4,$$
$$i_1 r_2 = i_4 r_3,$$

whence $$\frac{r_1}{r_2} = \frac{r_4}{r_3},$$

or $$r_1 : r_2 = r_4 : r_3.$$

SECTION 51.—**Joule's Law.**

We have seen that if the unit of positive electricity under the influence of electrical forces passes from an equipotential surface on which the potential has the value V, to a second equipotential surface on which it has the value V′, the work of the electrical forces is given by $V' - V$. Let us now again refer to the circuit in connection with which we (Fig. 86) explained Ohm's law, and let us consider in the first place the connecting wire D between the surfaces of contact A and B. If I is the strength of the current, we understand by this the quantity of electricity which passes through a cross-section of the conductor in a second. This quantity of electricity consists of two equal parts of positive and negative electricity, which flow in opposite directions. The quantity of positive electricity is $\frac{I}{2}$, and the amount of negative electricity is the same.

We have denoted the potential at A by V_{1d}, that at B by V_{2d}. Now if $\frac{I}{2}$ units of positive electricity flow from the position A to the position B, the work done is equal to $$(V_{2d} - V_{1d})\frac{I}{2}.$$

In an equal period of time, however, $\frac{I}{2}$ units of

negative electricity flow from the position B to the position A; the work which is hereby done has the magnitude

$$(V_{1d} - V_{2d}) \times -\frac{I}{2},$$

or

$$(V_{2d} - V_{1d})\frac{I}{2}.$$

The total work therefore done in the part of the circuit between A and B is equal to

$$(V_{2d} - V_{1a})I.$$

In every second $\frac{I}{2}$ units of positive electricity pass from A, and in every second there arrive just as many units of positive electricity at B, and the same is true, but in an inverse sense, of negative electricity. Therefore $(V_{2d} - V_{1d})I$ or $R_d I^2$ (p. 201) *is the work done by the electrical forces in the part* D *of the circuit in a second.*

The condition of the current is stationary—unchangeable as regards time; the work named produces no change in the condition—it is apparently lost. If we examine the circuit more narrowly, however, we find that it gradually becomes warmer. The work is therefore converted into heat.

The work done by the electrical forces in a second has, in the part Z of the circuit, the magnitude

$$(V_{2z} - V_{1z})I, \quad \text{or } R_z I^2.$$

In the part of the circuit F the magnitude

$$(V_{2f} - V_{1f})I, \quad \text{or } R_f I^2.$$

In the part of the circuit f the magnitude

$$(V_{2f'} - V_{1f'})I, \quad \text{or } R_{f'} I^2.$$

And in the part of the circuit K the magnitude

$$(V_{2k} - V_{1k})I = R_k I^2.$$

The work of the electrical forces done in the whole circuit during a second equals

$$(R_d + R_g + R_f + R_{f'} + R_k)I^2,$$

or, if we denote the total resistance of the circuit by R, and the electromotive force of the circuit by E, it equals

$$RI^2,$$

or (equation 12, p. 201) it equals

$$IE;$$

i.e. *it is equal to the product of the total resistance into the square of the strength of the current, or equal to the product of the strength of the current into the electromotive force of the circuit.*

This work is converted into heat. If we divide the amount of work by the mechanical equivalent of heat, we obtain the quantity of heat which appears in the circuit in every second.

This is Joule's law, which has been experimentally established by manifold researches.

Thin wires are much more quickly heated than thick ones, because the resistance of the former is greater. If we observe the temperature to which a wire of varying cross-section is heated in its different parts by one and the same current, it is to be noted that if, *e.g.* the cross-sections are circles, and if one has radius 1 and the other radius 2, the cross-sections are as 1 : 4, and therefore the resistances of two equal lengths of the conductor of these cross-sections are as 4 : 1. The masses of these parts of the conductor are in the ratio of 1 : 4 ; consequently the temperatures to which they become raised by equal quantities of heat are as 4 : 1. Now the quantities of heat which are engendered in the two wires by the current have the

same ratio as the resistances—therefore they are as 4 : 1 ; the temperatures, therefore, to which the parts of the conductor are raised by the current are as 16 : 1, or as $2^4 : 1^4$, or *inversely as the fourth powers of the radii.**

* This result is true only of the rates at which the temperatures of the two wires *begin* to rise. As soon as the current has passed for a finite time, the thin wire, being hotter than the thicker one, has a higher specific resistance, which tends to make its temperature rise at a still more rapid rate than that stated in the text. On the other hand, as the temperature rises there is a more and more rapid loss of heat from the surface. If specific resistance and the coefficient of surface-emission were constant, the final temperatures would be inversely as the *third powers* of the radii ; but, as so little is known of the values at high temperatures of the two factors just mentioned, the calculation is not of very much interest.—TR.

CHAPTER IV

MAGNETISM—ELECTRO-MAGNETISM—ELECTRO-DYNAMICS—AND INDUCTION.

SECTION 52.—**Magnetic Potential.**

SUCH ponderable bodies as possess the property of attracting and holding fast portions of iron are, as is known, called magnets. If a magnet is suspended in such a way that it can freely turn round a vertical axis, the one end always points to the north, the other to the south. The former end is named the north pole, the latter the south pole.

The north poles of two magnets repel one another, and the same is true of the south poles. On the other hand, if the north pole of the one is brought near the south pole of the other, the two attract one another. The north pole and the south pole of a magnet have exactly opposite characters, just as positive and negative electricity.

In order to explain magnetic phenomena the assumption is made that the carriers of the magnetic forces are certain kinds of matter, the so-called magnetic matter, and that the magnetic matter found at the north and south poles is of an opposite nature, just as positive and negative electricity.

Magnetic matter exhibits, as follows from Coulomb's experiments (1785), precisely the same law of mutual action as electricity—the force with which two portions of north magnetic matter repel one another, or with which north magnetic matter and south magnetic matter attract one another, acts, when the portions of matter are concentrated in points, in the straight line connecting the points, and is proportional directly to the product of the quantities, and inversely to the square of the distance.

As a unit of north magnetic matter—we will in what follows designate it briefly as unit of north magnetism—that which repels an equal quantity of north magnetic matter at unit distance with unit force will be chosen, and similarly for south magnetism. Since north magnetism is related to south magnetism, just as positive to negative numbers, the former will be expressed by positive and the latter by negative numbers, so that, *e.g.*, the unit of north magnetism will be denoted by $+1$ and the unit of south magnetism by -1. If we measure two quantities of magnetic matter which are concentrated at two points, and find for them the numbers μ_1 and μ_2, the force of the mutual action is expressed by

$$\frac{\mu_1 \mu_2}{r^2},$$

where r denotes the distance between them.

An essential difference between magnetic and electric matter is that, whilst the latter can pass from one body to another, in regard to the former this is not possible. There can be present on a body more positive than negative electricity, and conversely, but magnets always contain an equal quantity of each kind of magnetism. The proof of this follows very

simply from the following fact: The earth is, as known, a great magnet. Its magnetic forces at the different points of the surface have different values, but the change is so gradual that, *e.g.* at all points of a room we can suppose the magnitude and direction of the earth's magnetism (*i.e.*, of the force with which the earth attracts the unit of north magnetism towards the north) as uniform. If we fasten a magnet to a piece of wood and put it on the surface of water in a vessel, then if the piece of wood is selected so that the magnet can float freely on the surface, the north magnetism will be attracted by the earth towards the north, the south magnetism towards the south, that is, in an opposite direction. Now if the amount of the north magnetism in the magnet were greater than the south magnetism, then the magnet would be moved towards the north; and if it were smaller, motion would take place in the opposite direction. But neither of these results occurs. The only effect of the earth's magnetism consists in a rotation of the magnet.

The action of the earth's magnetic force is the action of a couple. If we turn a freely suspended magnet round an axis perpendicular to the earth's magnetic force from its position of equilibrium, then the moment of the couple will increase with the angle through which the magnet has been turned, and will reach its greatest value when this is equal to 90°. This maximum value is directly proportional to the magnetic force of the earth—it will be 2, 3, 4, . . . times as great when this becomes 2, 3, 4, . . . times as great; or, in other words, the quotient of the maximum value of the couple by the earth's magnetic force is a constant quantity for each magnet. This constant quotient is called the *moment of the magnet.*

If a magnet is broken into numerous small fragments, these severally appear as complete magnets. We are, therefore, led to the assumption that the smallest parts, the elements of the magnet, are complete magnets. If they have the same orientation in the magnet, then the moment of the magnet is equal to the sum of the moments of the elementary magnets. The moment of each elementary magnet is obtained by multiplying the quantity of its north magnetism into its length.

Since magnetic matter follows precisely the same law of mutual action as electricity, all that we have said about electrical potential will be true of magnetic potential without further modification. If there are magnets in space, the work which the forces proceeding from them will perform, if we bring the unit of north magnetism from infinity to a given point in space, is called the potential of these magnets, or the magnetic potential at the point referred to.

If only the unit of north magnetism were present in all infinite space this would, at a point which is distant r from it, produce a potential

$$-\frac{1.}{r}$$

If we replace the unit of north magnetism by the magnetic mass μ, this produces the potential

$$-\frac{\mu}{r};$$

and if we have given the masses μ_1, μ_2, μ_3, . . . (these being concentrated at points), their potential at a point which is distant r_1, r_2, r_3, . . . from them possesses the magnitude

$$-\left(\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}+\frac{\mu_3}{r_3}+\ldots\right).$$

In this expression the north magnetic masses are naturally expressed by positive, the south magnetic by negative numbers. At every point in space the magnetic potential has a definite value. Two different points have in general a different value of potential; the difference of this value is termed *the difference of magnetic potential at the points.*

Those surfaces whose points all possess the same potential are called equipotential surfaces. If we displace the unit of north magnetism on an equipotential surface, the magnetic forces do no work. Now since the magnetic forces are in general different from zero, it follows from this that they are perpendicular to the equipotential surfaces. If the unit of north magnetism can follow freely the action of magnetic force, it will move everywhere so that its path passes perpendicularly through the equipotential surfaces.* This path is called *a line of magnetic force.*

If we bring the unit of north magnetism from one point to a second point infinitely near, the potential in general changes. We obtain the component of magnetic force acting in the direction of the displacement if we divide the increase of potential by the amount of the displacement. If the quotient is positive the component acts in the same sense as the displacement, if it is negative it acts in the opposite sense. Now since the quotient is positive only when the increase of potential is positive, *i.e.* when the motion takes place from a place of lower potential to a place of higher

* Since the direction of motion of a material particle (point) does not in general coincide with the direction of the force, we must suppose, as regards the above motion of the unit of north magnetism, that velocity is always infinitely small, viz. as if the motion took place in an exceedingly resisting medium. Precisely the same is true regarding what has been said on p. 56.

potential, then the unit of north magnetism, if it can freely follow the magnetic forces, always moves so that it passes from places of lower potential to places of higher potential.

Just as we understand by the electrical force at a point the force acting on a unit of positive electricity at that point, so we also understand by the magnetic force at a point the force which is exerted on the unit of north magnetism at that point.

The space which surrounds a magnet is designated as the magnetic field of the magnet. The magnetic force in the different points of this field has in general a different magnitude and direction. If the strength of the magnetic field at a given point is spoken of, the magnitude of the magnetic force acting there is always understood.

SECTION 53.—**Magnetisation.**

When a conductor is brought into the neighbourhood of an electrical point, an electric separation takes place on it. Something precisely analogous manifests itself if we bring a piece of soft iron into a magnetic field. If we, namely, bring a piece of soft iron near the pole of a magnet, this will be attracted in the same way as the unlike pole of another magnet. And if we suspend the piece of soft iron by the one end on the pole, its other end will act as the pole of a magnet. We know this from its behaviour towards other magnetic poles, and because other pieces of soft iron remain suspended in contact with it. We can therefore say, the soft iron acquires in the neighbourhood of a magnet the property of a magnet, and we may form the conception that the smallest particles or elements of the

piece of soft iron are magnets which lie in the direction of the lines of force.

Faraday showed in the year 1845 that not only iron but all substances were attracted by a magnetic pole, and that this attraction becomes a repulsion when the medium surrounding the body undergoes a stronger attraction, bulk for bulk, than the body itself. We thus see that the property of being attracted by a magnet is common to all bodies, and that the action of a magnetic pole really depends on the surrounding medium.

Electrical influence affords us a ready means of making the direction of electrical forces evident. A short thin piece of wire freely suspended at its centre by an insulating silk thread takes such a position under the action of electrical forces, that its axis indicates the direction of the line of force passing through its centre. We can also make the direction of the magnetic lines of force completely visible if we take a piece of soft iron wire. Indeed on this fact the old and universally known process of making the magnetic lines of force visible by means of iron filings depends. If for instance a sheet of pasteboard is placed on the pole of a magnet, and is strewed over with iron filings, the so-called *magnetic curves* become recognisable under the influence of the horizontal components of the magnetic force.

SECTION 54.—**Magnetic Double Shell.**

Let two equal parallel surfaces (Fig. 91) be uniformly covered, the one with north magnetism, the other with south magnetism, and let the density on the former, or AA, be equal to 1, and on the latter, or BB, be equal to -1. By the density of the covering is understood, as

before, the mass which covers the unit of area. We will assume the distance of the two surfaces to be very small and denote this by d. How great is now the potential of this magnetic shell? The potential at the point C is, in conformity with the definition, that work which the magnetic forces of the double shell perform when the unit of north magnetism arrives at the point C from infinity. Since this work is independent of the path, we can assume, for the sake of simplicity, that the motion takes place on the straight line CD. By the principle of action and reaction, the force with which the double shell acts on the unit of north magnetism is exactly as great as the force with which this acts on the double shell. If we therefore imagine the unit of north magnetism held fast at the point C, and the double shell brought from infinity to its present position, during which we will assume for the sake of simplicity that it moved in the direction DC, so as always to remain parallel to itself, the work of the force of attraction of the unit of north magnetism will now be evidently equal to the former work, or equal to the potential of the double shell at the point C.

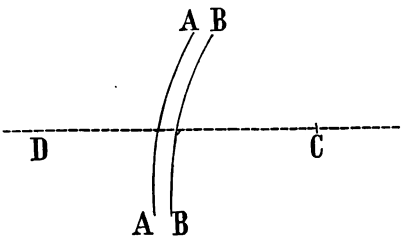

Fig. 91.

In order to ascertain the amount of this work we make use of the proposition given in section 19 and somewhat modified in section 27. We there arrived at the statement that, when we cover a closed surface uniformly with matter, so that the unit of mass is distributed over the unit of surface, the tension under which

the enclosed space stands is zero value when the masses acting are situated outside the surface. To this tension each portion of the closed surface contributes its share, and this consists of the force with which the mass situated on that portion of the surface is acted on in the direction of the normal. This share is (as in section 27) positive when the force acts outwards along the normal, in the opposite case it is negative.

The lines of force of the unit of north magnetism are, provided this is concentrated at a point, straight lines. If we draw through all points of the bounding curve AA the magnetic lines of force of the unit of N., we obtain a conical surface with the vertex at C. If further we describe with centre C and radius unity a spherical surface, we obtain an enclosed space which is bounded by the shell AA, by the conical surface, and the sphere. The shell AA is covered with north magnetic matter of unit density. If we now suppose the remaining bounding surfaces of the space under consideration covered uniformly with north magnetic matter in such a manner that the mass unity is distributed over the unit surface, then must the tension under which the space stands, in so far as this is derived from the unit of N., have the value zero. The covering of the conical surface contributes no share to this tension, because the direction of the magnetic force acting there lies in the surface. The sum of the forces with which the portions of the mass of the shell AA will be repelled by C in the direction of the normal will therefore be equal to the force with which the covering of the portion of the spherical surface is repelled. If this has the magnitude ω, then the mass on this is equal to ω, and consequently ω is the force with which this mass is repelled by C, because the radius of the sphere is equal to unity.

The potential of the double shell at the point C is compounded of the potential of the shell AA and of the shell BB. If we keep the unit of north magnetism fixed at C and bring the shell AA from infinity to its present position, then must the work of the magnetic force proceeding from C be equal to V. If we bring the shell BB from infinity to the position AA, the work of the force proceeding from C is equal to $-V$. If we now bring the shell BB from the position AA to the position BB, then the force components acting in the direction of the normal alone perform work on the separate parts. Each of these components acts along the distance d (the distance between the two shells), viz. in the direction of the motion, because the forces now act attractively. Now since the sum of all the components is equal to ω, the work done is now equal to ωd. We therefore obtain for the total work of the force proceeding from C, by the motion of the double shell from infinity to its present position, the expression,

$$V - V + \omega d,$$

or

$$\omega d.$$

The portion ω of the spherical surface is the measure of the solid angle under which an eye placed at C sees the double shell. We may therefore say: *The potential of the double shell at a given point is equal to the product of the distance between the two shells into the solid angle under which an eye placed at this point sees the double shell.*

If the density on AA equals ρ, and on BB equals $-\rho$, then the potential is ρ times as great, or equal to

$$\omega\rho d.$$

As is seen, the potential is positive. If the point C faces the north magnetic surface of the shell, instead of the south magnetic surface, the potential is negative.

For if ρ is taken negative, the surface AA is south magnetic, and BB north magnetic, and the expression for the potential is negative.

The potential at the point C does not change if the three quantities ρ, ω, d change in such a manner that their product $\rho\omega d$ remains constant. All double shells in regard to which the product ρd is the same engender at the point C the same potential, provided they subtend the same angle ω (Fig. 92).

Let us now consider a double shell, whose surfaces are flat (Fig. 93). Let AA be north magnetic and BB

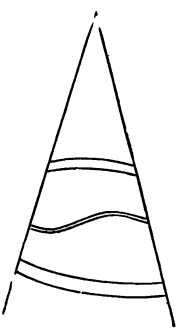

Fig. 92.

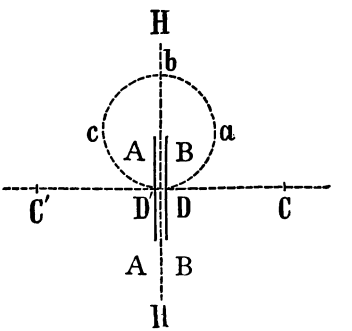

Fig. 93.

south magnetic. If the unit of north magnetism moves towards D on the straight line CD, the potential continuously increases, and at D is equal to $2\pi\rho d$. If, on the other hand, the unit of north magnetism moves from infinity on the straight line C'D' to the point D', the potential is in this case always negative, and at D' is equal to $-2\pi\rho d$. The domain of the positive potential values is separated from the domain of the negative potential values by the plane HH, which extends parallel to the surfaces of the shells and at equal distances from them ; for the potential on it has everywhere zero value,

since it is symmetrically situated with regard to the two faces of the shell.

If the unit of north magnetism moves along the straight line DD′ from D′ to D, then the potential changes continuously from the value $-2\pi\rho d$ through the value zero (on the plane HH) to the value $2\pi\rho d$. The value of the potential changes in this case by the amount $4\pi\rho d$. The same occurs if the unit of north magnetism goes from D′ to D along the path D′*cba*D.

The unit of north magnetism will, left to itself, under the influence of the forces proceeding from the double shell, always pass from places of lower potential to places of higher potential. If it is situated at the point *b* (on the plane HH), then it will move at right angles to HH (because this plane is an equipotential surface) from left to right. By the principle of the equality of action and reaction, we must conclude that, if the unit of N. is held fast at the point *b*, the several portions of each shell will be subject to forces equal and opposite to those which they exert on the unit, If the two surfaces of the shell are firmly joined together. the double shell experiences a displacement from right to left, and at the same time a movement of rotation, by which the double shell seeks to turn the south polar side to the unit of north magnetism.

Were, *e.g.*, the two surfaces circular, and could they turn round a fixed axis which bisects the straight line joining their centres and is perpendicular to the plane passing through the centres and the unit of N., then the double shell would revolve until the unit of N. was in the straight line joining the centres, and was opposite the south polar side. The potential has in this case reached its greatest possible value.

SECTION 55.—Electro-magnetic Potential.

In the year 1819 Oerstedt, the Danish philosopher, made the discovery that galvanic currents also exert magnetic force at a distance.

The law according to which the galvanic current deflects a magnetic needle was expressed by Ampère in a simple manner in the rule called after him. It is as follows: *If a swimmer be imagined swimming in the direction of the current and having his face turned to the needle, the north pole of the needle is deflected to the left hand.*

If a galvanic current (Fig. 94 *a*) flows perpendicularly through the plane of the paper, outwards towards the reader, the current seeks to turn the needle *ns* so that the direction *ns* runs perpendicularly to the direction of the current, and the two poles *n* and *s* are equally distant from the current. If we have several needles in the neighbourhood of the current, they all experience rotations in the same sense.

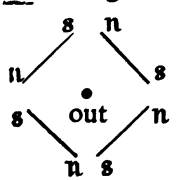

Fig. 94 *a*.

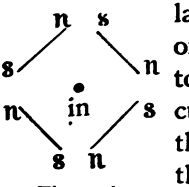

Fig. 94 *b*.

If the current proceeds in the opposite direction, as in Fig. 94 *b*, if it therefore passes through the plane of the paper from above to below, the needles will assume reversed positions.

When the current is made to pass perpendicularly through a piece of cardboard (Fig. 95), and the paper is strewed with iron filings, the separate particles of iron are magnetised by the current, and take up such positions as would be assumed by small

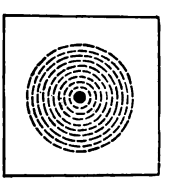
Fig. 95.

magnetic needles. A representation of the magnetic lines of force, or of the magnetic curves of the current, is in this way obtained. Fig. 96 gives an approximate idea of the magnetic effect of the whole circuit.

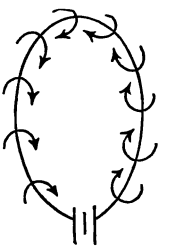

Fig. 96.

In order to obtain the magnetic potential, or, better stated, the electro-magnetic potential of the current, the assumption is made that the magnetic forces of the current everywhere outside the current follow the same law as if they proceeded from magnetic matter. In this manner, with the help of the higher calculus, the result is obtained that the magnetic effect of the current at a distance is exactly the same as if *it came from a magnetic double shell* which is bounded by the conductor of the current, and whose north polar face possesses the density $\frac{kI}{4\pi d}$, and whose south polar face has the density $-\frac{kI}{4\pi d}$, where I denotes the strength of the current, d the distance between the shells, and k a constant. In connection with this there is still one thing to be noted. The magnetic double shell takes the place of the current as regards its magnetic effect at a distance at all points of space with the exception of the points situated between the faces of the shell.

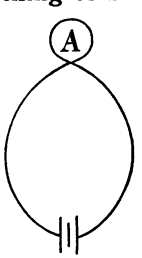

Fig. 97.

If the conductor of the current forms a loop (Fig. 97, A), this likewise acts at a distance just as a magnetic double shell which is bounded by it. If the shell is flat, and if f is the area of the surface which is enclosed by the loop, there is on the north polar face

the quantity $\frac{fk\mathrm{I}}{4\pi d}$, and on the south polar face the quantity $-\frac{fk\mathrm{I}}{4\pi d}$. The magnetic moment of this double shell is therefore equal to

$$\frac{fk\mathrm{I}}{4\pi d}.d,$$

or

$$\frac{fk\mathrm{I}}{4\pi}.$$

Now if f is equal to the unit surface (one square centimetre) and I is of the magnitude

$$\mathrm{I} = \frac{4\pi}{k},$$

the moment is then equal to 1. The strength of the current $\mathrm{I} = \frac{4\pi}{k}$, which satisfies this condition, we will select as the unit strength of current, and consequently put it equal to 1. Under these circumstances k is determined by the equation

$$k = 4\pi.$$

The new unit of current is called the electro-magnetic unit of current. This is, as we see, that current which, flowing round the unit of surface, produces at a distance the same magnetic effects as a short magnet of moment unity standing perpendicular to the surface. By the introduction of this new unit of current the electro-magnetic potential acquires the expression (*cf.* section 54)

$$\pm\frac{\mathrm{I}}{d}\omega d,$$

or

$$\pm\mathrm{I}\omega,$$

where ω has the same meaning as before. Since the potential of the imaginary shell is independent of the distance d, we can consider d to be infinitely small or

that the faces coincide. It follows then that their density is to be considered as infinitely great.

With reference to the magnetic double shell which takes the place of the current, the following is to be specially observed: The single shell is bounded by the current conductor. This possesses two faces. That face is named positive whose normal stands in such a relation to the direction of the current that if any one places himself on the surface in the direction of the normal, and traverses the boundary in the direction of the current, the surface always remains on the right hand. The magnetic double shell is now so constituted that one surface bounded by the conductor of the current possesses north magnetism of the density $\frac{I}{d}$, and a second surface equal in extent lying on the positive side of the former, and parallel to it at distance d, possesses south magnetism of density $-\frac{I}{a}$.

The relation between the direction of the current and the position of the surfaces can also be expressed in the following manner: If a swimmer imagines himself swimming in the direction of the current, and having his face turned towards the shell bounded by the current, the south magnetic face lies on his right hand.

If the current turns the positive side to the unit of N. the potential is positive, in the opposite case negative, or in other words: *If an eye placed at the position of the unit of* N. *sees the current circulating in the same way as the hands of a watch, the potential is then positive, in the opposite case negative.*

If we construct a cone with a given point as vertex, then all galvanic currents of equal strength flowing in circuits which lie in the surface of the cone, however

their course may otherwise vary, produce the same electromagnetic potential at the vertex of the cone, provided that an eye situated there sees all the currents circulating in the same direction. If two such currents have opposite directions, then they together engender zero potential at the vertex of the cone.

SECTION 56.—**Motion of Magnets and Currents.**

We will now for the sake of simplicity consider a plane circuit, *i.e.* a conductor of a current whose parts all lie in one plane (Fig. 98). Let the current have the strength I and let its direction be that denoted by the arrow. Let the straight line XX′ cut the plane of the current at the point C. Now if the unit of N. is brought from infinity to C along the straight line CX, the potential continuously increases from zero, and at C will be equal to $2\pi I$. If, on the other hand, we bring the unit of N. from infinity to C along the straight line X′C, then the potential will change from the value zero through values that are always negative till at C it is equal to $-2\pi I$. We see from this that the potential at the point C is either $2\pi I$ or $-2\pi I$, according as the unit of N. has approached from infinity on the course CX or X′C; in other words, *the potential at the point* C *is capable of several values.*

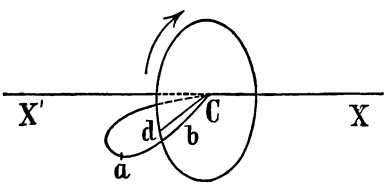

Fig. 98.

The point C is not the only point having this property; the potential at all points of the plane of the current within the conductor of the current is capable of several values. Herein lies the real distinction between

the electro-magnetic potential of a current and the magnetic potential of a magnetic double shell of finite density.

If we bring the unit of N. from the point C, after it has been brought there from infinity along X'C, by the path C*ab*C to C again, the potential passes from the value $-2\pi I$ through the value zero to the value $+2\pi I$. At the point *a*, at which the path cuts the plane of the current, the potential has the value zero, for an eye at *a* sees the current under the angle zero. Before arriving at the point *a* the potential had negative values; beyond the point *a* it had positive values. During this motion the potential increases to the extent of $2\pi I - (2\pi I)$ or $4\pi I$. Let this motion take place once more, and the unit of N. pass again through *a* and *b* to C, the potential once more increases to the extent of $4\pi I$, and so on.

If the unit of N. can freely follow the magnetic forces of the current, it moves in such a manner that the potential continuously increases from lower values to higher values. Were the unit of N. fixed at the end of the arm C*d*, which can turn freely and easily at the point *d* around the conductor of the current, this arm would experience a continuous rotation in such a way that the unit of N. would describe the path C*ab*C.

In reality the experiment cannot be carried into effect so simply, because the unit of N. cannot be isolated. The magnetism, as we have seen, is so distributed in every part of the magnet that the north magnetic and south magnetic matter occur in equal amounts. The rotation can, however, be brought about by a simple modification. For instance, at the point *a* the unit of N. traverses the plane of the current in an opposite sense to that in which it traverses it at the point C; it follows from this that the unit of south magnetism brought to *a* must move in the same

direction through the plane of the current as the unit of N. at the point C.

If the north pole of a magnet is at the point C and the south pole of a magnet at the point *a*, the magnet is acted upon at both poles by forces perpendicular to its axis and in the same direction. In Fig. 99 we see these relations delineated. The current enters by the binding screw P, passes through the wire *f* into the

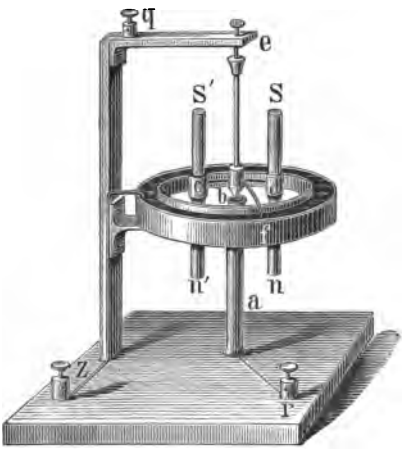

Fig. 99.

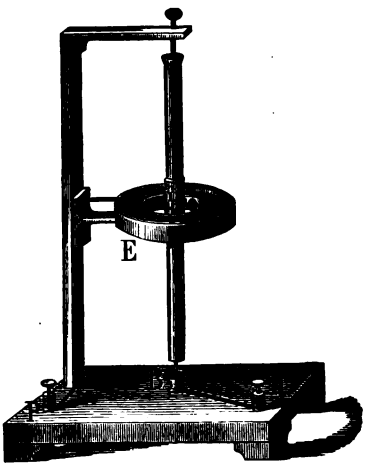

Fig. 100.

mercury in the channel, by the wire on the left to the binding screw Z, and thence back to the battery. Or, the current may enter by the binding screw *q*, pass down the rod *eb*, and then to Z and the battery as before. The forces on the north and south poles of the left-hand magnet and that on the north pole of the right-hand magnet are in the same direction ; they will therefore strengthen each other's effect. The force on the south pole of the right-hand magnet is indeed in the opposite sense, but it is smaller than each of the three

others, and is consequently greatly surpassed by their sum. Rotation follows, as seen from above, opposite to the hands of a watch.

We may use, as in Fig. 100, the magnet itself from the one pole to its centre as a conductor of a current, and if the magnet can turn easily round its axis, a rotation likewise ensues, which, seen from above, is opposite to the motion of the hands of a watch. The explanation is quite simple. We may consider the magnet as if it were made up of a large number of thin magnetic rods of the same length and similarly placed. If we imagine the two magnets in Fig. 99 to be put together so that the axis of rotation lies in the plane of mutual contact, the motion described must still take place in this case; it must also take place if we add to the two magnets several magnets in addition having the same length and the same orientation. Such a bundle of magnets may represent a single magnet revolving round its own axis.

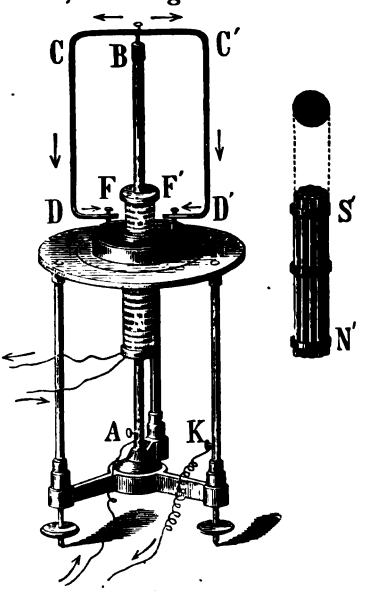

Fig. 101.

Just as in the last case the fixed current turns the magnet about its axis, so by the principle of the equality of action and reaction the current must rotate, if it is capable of being turned, as soon as we hold the magnet fixed (Fig. 101. Compare with this Fig. 100).

SECTION 57.—**Tangent Galvanometer.**

If we assume that all portions of the current participate in the magnetic force of the galvanic current, and if the force is distributed among the various parts, the magnitude of the force with which a portion of the current of the length l acts on the unit of N. becomes

$$I\frac{l}{r^2}\sin(l, r).$$

In this I denotes the strength of the current, r the distance of the unit of N., and (l, r) the angle formed by l and r. The direction of the force is perpendicular to that plane, which can be placed through the unit of N. and the portion l of the current; viz. towards that side on which an observer, standing upright and turned towards the current, sees the current flowing past from right to left; in other words, Ampère's rule holds good.

We will at once apply this law to an important case. Let the unit of N. be situated in the centre of a circular current (Fig. 102). This is at a distance from all parts of the current equal to the radius of the circle, and therefore equally distant. Let us call the radius r. All parts of the current make the same angle with r, viz. the angle 90°, whose sine is equal to 1. All parts of the current, therefore, which have the same length, exert the same magnetic force at the centre, viz. the force

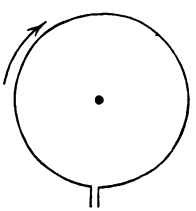

Fig. 102.

$$\frac{Il}{r^2},$$

where l denotes the length of the parts of the current.

Since this force, in accordance with Ampère's rule, has the same direction for all the parts of the current, the magnetic forces of all parts of the current are added, and a total force given whose amount is

$$\Sigma \frac{Il}{r^2},$$

where Σ represents the recognised symbol of summation. Now all the terms of the sum have the factor $\frac{I}{r^2}$ common ; we can therefore take it out, and obtain

$$\frac{I}{r^2} \Sigma l,$$

or, because Σl, the sum of all parts of the conductor, is the circumference of the circular current,

$$\frac{I}{r^2} \cdot 2\pi r,$$

or

$$\frac{2\pi I}{r}.$$

If there are at the centre m units of N., these will be acted on by the force

$$\frac{2\pi m I}{r},$$

perpendicular to the plane of the current. And if there are at that place m units of south magnetism, the force is the same but opposite in direction. If we bring to the centre a very short magnet whose length is λ, and whose north pole has the strength m, and whose south pole possesses the strength $-m$, and if the axis of the magnet lies in the plane of the current, a couple acts on the magnet, the moment of the couple being

$$\frac{2\pi m \lambda I}{r}.$$

$m\lambda$ is the moment of the magnet: if we call this M, then the moment of the couple takes the expression

$$\frac{2\pi MI}{r}.$$

This couple causes a deflection of the magnet. If in consequence it forms with the plane of the current the angle a, the moment of the couple has become smaller and equal to

$$\frac{2\pi MI}{r}\cos a,$$

because the perpendicular distance of the two forces of

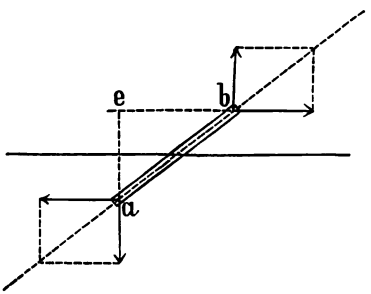

Fig. 103.

the couple is equal to $\lambda \cos a$ (Fig. 103). The greater a becomes the smaller cos a becomes, the smaller also the moment of the couple.

If the plane of the current is parallel to the direction of the earth's magnetic force, or in the plane of the magnetic meridian, the earth's magnetic force seeks to retain the magnet in the plane of the current. If the magnet can now turn round a vertical axis, only the horizontal component of the earth's magnetic force comes into play, and this—we name it H (per unit of N.)—engenders a couple whose moment is (Fig. 103)

$$m\mathrm{H} \,.\, ae,$$

or $$m\mathrm{H} \cdot \lambda \sin \alpha,$$

or $$\mathrm{MH} \sin \alpha.$$

This moment increases with the angle α, and acts in an opposite sense to the moment due to the magnetic force of the current. If the magnet is therefore turned by the current out of the plane of the current, the moment of the magnetic force of the current becomes smaller, whilst that of the earth's magnetic force increases continuously from zero; at a certain angle the two will be equal to one another, and will mutually neutralise one another. If we name this angle α we obtain the equation

$$\frac{2\pi \mathrm{MI}}{r} \cos \alpha = \mathrm{MH} \sin \alpha,$$

or $$\tan \alpha = \frac{2\pi \mathrm{I}}{r\mathrm{H}}.$$

The angle α is, as we see, entirely independent of the magnitude M, the moment of the given magnet.

If H is known and r determined, we can with the help of this equation determine I, the strength of the current. It is namely

$$\mathrm{I} = \frac{r\mathrm{H}}{2\pi} \tan \alpha,$$

or I, *the strength of the current, is proportional to the tangent of the angle of deflection.* On this proposition rests the principle of the tangent galvanometer.

Since the calculation is only correct when the two poles are very near the centre, it follows from this that only very short magnets can be employed.

If several turns are taken, all will act in the same way. If their number is n, the moment of the magnetic force proceeding from the current is equal to

$$\frac{2\pi \mathrm{M}n\mathrm{I}}{r} \cos \alpha.$$

The moment of the earth's magnetic force is equal to

$$HM \sin a.$$

When both are in equilibrium, then the equation

$$\frac{2\pi MnI}{r} = HM \sin a$$

applies,

or $$\tan a = \frac{2\pi nI}{rH},$$

or $$I = \frac{Hr}{2\pi n} \tan a.$$

SECTION 58.—Magnetic Force of a Coil.

Let us now consider a long coil whose turns are all circular, and let them have the same radius. Let the number of turns be n. If the turns lie near each other, their magnetic effect may be regarded in the same way as if n circular currents were arranged side by side parallel to each other, and at the same distance apart as the turns, therefore as represented in Fig. 104. Our task will now be to ascertain the magnetic force of the coil at a point of the axis, for example at the point C. We assume that the current, as seen from the point C, circulates in the direction of the hands of a watch.

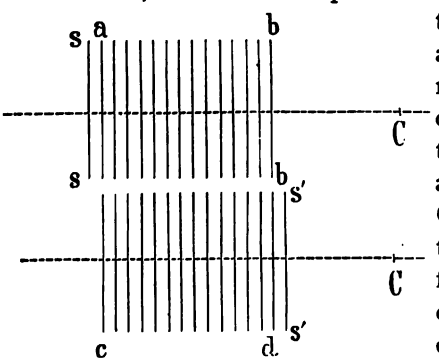

Fig. 104.

If I is its strength and ω the solid angle under which one of the circular currents is seen, the potential of this

current at the point C is equal $I\omega$, and the potential of all circular currents is the sum of these products, or $I\Sigma\omega$. We can separate the sum $\Sigma\omega$ into two parts, namely into ω_s, *i.e.* the solid angle under which the current *ss* is seen from C, and into that part of the sum which belongs to the other circular currents between *a* and *b*, and which we will denote with $\sum_a^b \omega$.

The force with which the coil attracts the unit of N. at the point C lies in the axis, because everything is symmetrical round this. In accordance with the principle of the equality of action and reaction, the unit of N. at the point C must attract the coil with the same force. How great now is this force?

Let us now imagine the unit of N. held fast at the point C, and the coil to be free to follow its magnetic force and to be displaced a distance equal to that between the circular currents—we will name it *d*. Then the potential of the coil at the point C is changed by this displacement, and in the same way the potential of the unit of N. on the coil. The change of potential divided by the amount of displacement *d* gives the force with which the coil is attracted. In order to ascertain the potential after the displacement we again have to multiply I, the strength of the current, by the sum of the solid angles under which the circular currents are seen from C. We can again as before separate the sum into two parts, into ω'_s, *i.e.* the solid angle under which we see the circular current *s's'*, and into that amount which relates to the other circular currents between *c* and *d*, and which we will denote by $\sum_c^d \omega$. Now, however, it is evident (cf. Fig. 104) that $\sum_c^b \omega$ is the same thing

as $\sum_{c}^{d}\omega$, for the entire displacement has had the same effect as if the last circular current *ss* had been taken away from its place and had been brought to the position *s's'*. The change of potential is therefore equal to

$$I\omega'_s - I\omega_s,$$

or

$$I(\omega'_s - \omega_s),$$

and consequently

$$I\frac{\omega'_s - \omega_s}{d}$$

is the force with which the coil is attracted by the unit of N., or conversely, the force with which the coil attracts the unit of N. at the point C.

If *d* is very small we can without further consideration understand ω'_s to be the solid angle under which the circular current *b* is seen. We therefore obtain the *magnetic force of the coil at the point* C *if we subtract the solid angle, under which we see the end turned away from us, from the solid angle under which we see the end turned towards us, divide this difference by the distance between the turns, and multiply the quotient by the strength of the current.*

We can still give the expression for this force a somewhat different form. Let R be the radius of the circular turns, and *r* the distance of the point C from *s*, the conductor of the current, and let us describe from the point C with radius *r* a spherical surface, then that part of it, or the spherical cap which is bounded by the conductor of the current, possesses

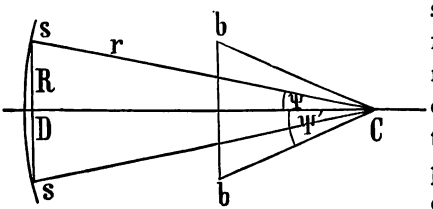

Fig. 105.

the area $2\pi r(r - \sqrt{r^2 - R^2})$ (Fig. 105), as can be proved by elementary methods. Now ω_s is that part of the spherical surface described with C as centre and radius 1, and which is bounded by the cone C*ss*. ω_s will therefore have the same ratio to the whole spherical surface of radius 1, or to 4π, as the spherical cap *ss* has to the whole spherical surface of radius r or $4\pi r^2$, *i.e.* in symbols,

$$\omega_s : 4\pi = 2\pi r(r - \sqrt{r^2 - R^2}) : 4\pi r^2,$$

from which

$$\omega_s = 2\pi\left(1 - \frac{\sqrt{r^2 - R^2}}{r}\right)$$

follows.

Now $\sqrt{r^2 - R^2}$ is nothing else than CD, therefore

$$\sqrt{\frac{r^2 - R^2}{r}} = \frac{CD}{Cs} = \cos\psi.$$

From this the equation

$$\omega_s = 2\pi(1 - \cos\psi)$$

follows.

In exactly the same way we obtain the equation

$$\omega'_s = 2\pi(1 - \cos\psi'),$$

and therefore

$$\omega'_s - \omega_s = 2\pi(\cos\psi - \cos\psi').$$

If we substitute this in the expression $I \cdot \frac{\omega'_s - \omega_s}{d}$, there results for the magnetic force of the coil at the point C the value

$$2\pi I \cdot \frac{\cos\psi - \cos\psi'}{d},$$

or if we denote the length of the coil by l, and put

$$l = nd,$$

the value is

$$2\pi n I \frac{\cos\psi - \cos\psi}{l}$$

This will be greater the nearer the unit of N. is moved towards the coil, for the cosine becomes indeed smaller when the angle becomes greater, but the angles ψ and ψ' increase during the approach in different degrees, viz. ψ' considerably more than ψ.

If the unit of N. moves up to the plane of the circular current b (Fig. 106), then $\psi' = 90°$, and $\cos\psi' = 0$. The magnetic force is then equal to

$$2\pi n \mathrm{I} \frac{\cos\psi}{l},$$

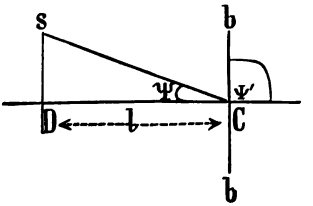

Fig. 106.

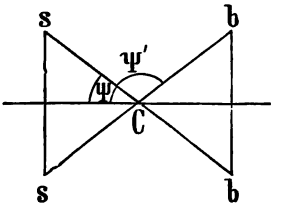

Fig. 107.

or, because (Fig. 106) $l = s\mathrm{C}\cos\psi = \sqrt{\mathrm{R}^2 + l^2}\cos\psi$,

$$\frac{2\pi n \mathrm{I}}{\sqrt{\mathrm{R}^2 + l^2}}.$$

If the unit of N. under the influence of this force moves into the coil, then ψ' becomes an obtuse angle, and consequently $\cos\psi'$ negative. If it comes to the middle of the coil (Fig. 107), then

$$\cos\psi = \frac{l}{2s\mathrm{C}} = \frac{l}{2\sqrt{\mathrm{R}^2 + \frac{l^2}{4}}},$$

and $$-\cos\psi' = \cos\psi,$$

consequently the force is equal to

$$\frac{2\pi n \mathrm{I}}{\sqrt{\mathrm{R}^2 + \frac{l^2}{4}}}, \quad \text{or} \quad \frac{4\pi n \mathrm{I}}{\sqrt{4\mathrm{R}^2 + l^2}}.$$

This force has the same direction as before; it is, however, greater than in the plane of the circular current *b*. If the unit of N. comes to the plane of the turn *ss* (Fig. 108), then $\psi = 90$, $\cos\psi = 0$, and

$$-\cos\psi' = \frac{I}{Cb} = \frac{I}{\sqrt{R^2 + l^2}}.$$

The force therefore acquires the value

$$+\frac{2\pi n I}{\sqrt{R^2 + l^2}}.$$

It is of the same magnitude and in the same direction as in the plane of the first turn, which is at once evident. If the unit of N. moves farther along the axis, the force has always the same direction, but its magnitude becomes smaller. The magnitude at points of the axis on the left of the turn *ss* will follow exactly the same laws as at points of the axis on the right of *bb*, so that at two points which are situated at the two sides of the coil at equal distances from it, it has the same value.

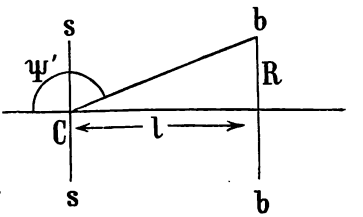

Fig. 108.

If the unit of N. passes along the axis of the coil from infinity on the right, through the coil again to infinity on the left, the force of attraction continually increases from zero, obtains at the middle of the coil its greatest value, decreases again from there, and at infinity again becomes equal to zero.

When the unit of south magnetism moves on the axis, the force has everywhere the same value as before, but its direction is opposite.

If we bring into the coil a rod of soft iron, this

becomes magnetised. The relation between the magnetic moment of the rod and the product $n\mathrm{I}$ is expressed approximately by the empirical formula given by Müller,

$$n\mathrm{I} = a\rho^3 . \tan \frac{\mathrm{M}}{b\rho^2},$$

in which a and b denote constant quantities, ρ the diameter, and M the magnetic moment of the rod. If the strength of the current becomes infinitely great, then is

$$\frac{\mathrm{M}}{b\rho^2} = \frac{\pi}{2},$$

or

$$\mathrm{M} = \frac{b\rho^2\pi}{2}.$$

The magnetic moment indeed increases with the strength of the current, but it approaches a limiting value as this becomes greater and greater.* The magnetised iron rod strengthens the intensity of the magnetic field of the coil to a very considerable extent. Experiments have shown that the intensity can become about thirty-three times as great. It is manifest that a decided advantage accrues from this.

SECTION 59.—Potential of two Currents on one another, and the Potential of a Current on itself.

The electro-magnetic potential of a current at a given point in space is, as we have seen, that work which the magnetic force of a current performs in moving the unit of N. from infinity into the given point. If we bring to this point m units of N., the work is then m times as great, or equal to m-fold the potential at the given point. If we keep the current fixed and bring a magnet from

* More recent experiments do not give any clear indication of a limit of possible magnetisation.—TR.

infinity into its neighbourhood, the magnetic forces of the current perform an amount of work whose magnitude we obtain if we mentally subdivide the magnetic matter of the magnet into very small parts and multiply each one of them into the potential of the current applicable to it, and add all the products. This sum is named *the potential of the current on the magnet.*

If the magnet is held fast and the current brought from infinity to its present position, the same amount of work is done; the potential of the magnet on the current is equal to the potential of the current on the magnet.

Something quite similar is true in regard to two currents. We have seen that we can consider every current as regards its external magnetic effects as a magnetic double shell. If we hold a current in a fixed position,—let its strength be denoted by I,—and if we bring a second current of the strength I′ from infinity into its neighbourhood, the magnetic forces of the former will perform work. We again obtain the amount of this if we mentally break up the magnetic double shell which takes the place of the second current into innumerable portions, and multiply each of them by the potential of the first current applicable to it, and add together the products. Now the potential of the first current is given by $\pm I\omega$, where ω denotes that solid angle under which the current is seen from the point in question, and the + or − sign holds according as the direction of circulation of the current for an observer is in accordance with the motion of the hands of a watch, or the opposite. The potential of the first current is therefore everywhere proportional to I, the strength of the current. The density of the double shell, by which the second current is represented, is likewise proportional to the

strength of the current I'. Now, because the numerical values of the portions into which we conceive the double shell representing the second circuit as being broken up are given by the product of the magnetic density of each part into the respective area, the factor II' will appear in all the products which are formed from the potential of the first current and the portions of the double shell in question. If we take out this from all the products and denote the sum of those remaining by Q, we get for the potential of the first current on the second current the expression II'Q.

The potential of one current on a second current—this is designated the electro-dynamic potential of the currents on one another—*is therefore directly proportional to the product of the strengths of the currents.* The factor Q depends merely on the form, the magnitude and relative positions of the conductors carrying the currents.

The potential II'Q is positive when the currents circulate in the same sense in both circuits—that is, keeping to the conception of the double shells, the north magnetic face of one double shell stands opposite to the south magnetic face of the other, otherwise it is negative.

If the currents freely follow the forces due to their mutual action, they move in such a way that the potential rises from lower values to higher. The absolute value of Q diminishes as the distance between the currents increases, and becomes nothing when the distance is infinite. If the product II'Q is positive, then Q is also positive, and the two currents will approach or will attract one another. If on the other hand the product II'Q is negative, the two currents mutually repel one another. In other words: *Currents having*

the same direction attract each other; currents having opposite directions repel each other.

If two currents change their relative position, then, as already stated, the electro-dynamic potential is changed. If the factor Q thereby acquires the value Q_1, and the strength of the current remains unaltered, the electro-dynamic potential of the currents in the second position acquires the value

$$II'Q_1,$$

and then

$$II'(Q_1 - Q)$$

denotes the work of the forces with which the currents act on one another. This work is designated the *electro-dynamic work* of the currents.

If the currents return to their original position, and if the strength of the currents has not changed during the motion, the electro-dynamic potential acquires the same value as before, and the *total electro-dynamic work is equal to zero.*

Just as a heavy sphere, magnet, and so on, possess a definite power of doing work (energy), so this is true regarding every individual galvanic circuit. In order to find the value of this work let us again have recourse to the magnetic double shell, and divide this mentally into innumerable small parts. If we multiply each of these by the value of the potential of the current applicable to it, and if we effect the summation of all these products, then the half of this sum gives us the value of the work sought, or *the potential of the current on itself.*

Since the potential of the current, as well as the value of every portion of the magnetic shell, is proportional to I, the strength of the current, all the above-mentioned products must contain the factor II or I^2,

and consequently the potential of the current on itself will be of the form qI^2,
where q again denotes a quantity which depends on the form of the conductor.

If two currents are in presence of one another, the total value of the work of this system is given by an expression of the form

$$II'Q + I^2q + I'^2q'.$$

SECTION 60.—**Induction.**

In discussing electro-dynamic work we made the assumption that the strength of both currents remains the same. In reality this assumption does not in general hold good. If the two currents change their relative positions, the strengths of the currents, as Faraday first showed (1831, 1832), also change. In addition to the given currents there appear new currents whose electromotive force depends upon the change of the relative positions of the currents. These new currents are named *induced currents.*

If we change the relative positions of two currents by starting from definite initial positions and returning to them again after a series of changes, the currents also at the end of this cycle of operations possessing the same strengths as at the beginning, and if we then calculate the total electro-dynamic work of the currents during the motion, the result no longer comes out zero. Were the electro-dynamic work the only work which is done in such a case, we could in this manner gain an unlimited amount of work. The electro-dynamic work is however not the only work, for, as has been already stated, electromotive forces are excited in consequence of the change of the relative positions of the currents,

and these likewise do work. Now F. Neumann having started with the assumption that the total work done by the cycle of operations must be zero, succeeded in ascertaining the magnitude of the electromotive force of the induced currents.

If I and I′ denote the strength of the current in the first and second circuits respectively, and if II′Q is the electro-dynamic potential of the currents on one another at a given instant, then the electromotive force of the induced current in the first circuit *will be given by the rate at which the quantity* $(-I'Q)$ *is changing at that instant. The electromotive force of the induced current appearing in the second conductor will likewise be given by the rate at which the quantity* $(-IQ)$ *is changing at the same instant.*

The following conclusions result from this: If two currents have the same direction, the electro-dynamic potential II′Q is positive, then Q is also positive. If the second current is now brought near to the first, Q becomes greater. Therefore $(-I'Q)$ becomes smaller—the rate of the change of $(-I'Q)$ is therefore negative; the electromotive force of the induced current is opposite to that of the given current I—consequently the induced current itself is in an opposite direction. We therefore arrive at the following proposition: *If a current is brought near to a circuit carrying a current, a current in an opposite direction is induced in the latter.* The removal of the second current from the first has the opposite effect. Q then becomes smaller, and in consequence $(-I'Q)$ becomes greater; the rate of change $(-I'Q)$ is now positive, the electromotive force induced in the first conductor is in the same direction as that producing current I, and consequently the induced current itself is in the same direction. Proposition: *If*

a current is removed from a circuit carrying a current, a current in the same direction is induced in the latter.

Both these propositions must be true whatever value I, the given strength of current, may have in the first current; they are therefore true when no current at all flows in the first conductor.

If we keep the current (I') at rest and move a closed conductor towards it, this is evidently the same if the conductor were kept at rest and the current moved. The same laws therefore hold for the approach and removal as before.

Not only the alteration of the relative positions of two currents, but also a change of the strengths of the currents, the relative positions remaining the same, produces induction. The same laws as above apply in this case for the electromotive force of the induced current. If the given currents have the same direction, II'Q is then positive, therefore Q is also positive. The quantity Q does not change when the relative positions of the currents remain the same. Now when I' becomes greater, then $(-I'Q)$ becomes smaller; the rate of change of $(-I'Q)$ is negative, consequently the current induced in the first conductor is in an opposite direction to the given current I. *If therefore the strength of a current becomes greater, a current in an opposite direction is induced in a closed circuit.*

When I' becomes smaller, then $(-I'Q)$ becomes greater; the rate of the change of $(-I'Q)$ is positive, and consequently the induced current has the same direction as the given current. *When the strength of a current becomes smaller, a current in the same direction is induced in a closed conductor.*

It is self-evident that the case of the opening and closing of a current is included in these two propositions.

The following case is also included in them: Let two completely equal currents, parallel to one another, be placed opposite. If we open both simultaneously, an induced current is engendered in each conductor such that these induced currents have not only the same direction themselves, but also the same as the original currents. If we again close the two circuits simultaneously, there arise in both conductors induced currents which again have the same direction themselves, but flow in a contrary direction to the original currents.

It is clear that no change takes place as regards the circumstances, if we bring both currents nearer one another, and finally cause them to coincide. They then form a single current of double strength. If we stop this current, an induced current takes place in the conductor in the direction of the given current; if we close it, an induced current takes place in the opposite direction. This case of induction is named *self-induction*, and the induced current the *extra current*.

If a magnet is moved in presence of a conductor, so that it approaches or recedes, or if magnetism appears or disappears in a rod in the neighbourhood of a closed conductor, an induced current occurs in the conductor. This induction, for the purpose of distinguishing it from the former, which is designated by the name *voltaic induction*, is named *magneto-electrical* induction.

The laws of magneto-electrical induction can be assimilated to those of voltaic induction in a very simple manner. Galvanic currents, for instance, exhibit magnetic action at a distance just as magnetic matter, only with the difference that the magnetic potential of given masses is always at every point susceptible of but one value, but that of a galvanic current at certain points of space is susceptible of several values. The galvanic

current, as far as regards its external magnetic effects, can indeed be represented by a magnetic double shell; but this does not hold for all points of space—the points within the double shell are excepted. It follows from this that it is not possible by any distribution of magnetic material whatever to produce a system which corresponds at all points of space with the electrical current. The converse is quite possible. It is always possible by a proper arrangement of infinitely small electrical currents to form a system that corresponds in every respect with the magnetic system. On this principle Ampère explained the action of magnets by means of molecular electrical currents. Currents are supposed to circulate round the individual molecules of a magnet, which currents all have the same direction, viz. such a direction that Ampère's swimmer would have the north pole of the magnet on his left.

Magneto-electrical induction is thus reduced to voltaic induction. *If a magnet is brought near to a closed conductor there arises in the conductor an induced current in a direction opposite to Ampère's molecular currents in the magnet. If the magnet is removed the induced current has then the direction of the molecular currents.* The same thing naturally holds true if we keep the magnet at rest and move the conductor towards it.

If the magnetism of a rod appears or disappears (by our closing or opening the current of a magnetised spiral), a current is induced in the conductor, the direction of which in the first case is opposite to that of the molecular currents, in the second case in accordance with it. The appearance and disappearance of the magnetism has the same effect as the bringing near and removing a magnet.

If a closed conductor moves in a magnetic field

produced either by a current or by a magnet, an induced current occurs in it. It is important to ascertain the amount of the force which acts at each point of the conductor on the unit of positive electricity. The application of the higher calculus leads, with the help of F. Neumann's law, to the result that the magnitude of this force *is given by the product of the strength of the magnetic field and the component of the velocity of the point perpendicular to the magnetic line of force which passes through the point.* The direction of the force *is perpendicular to the plane which contains the direction of the magnetic line of force at the point and the direction of the motion.*

If we fix our attention on the components of the velocity perpendicular to the magnetic line of force, we obtain three directions which are perpendicular to each other—the direction of the line of force, the component of the velocity, and that of the electrical force. *The direction of the electrical force is such that if one stands in the direction of the magnetic line of force* (so that the direction of the line is from foot to head), *and looks straight out in the direction of the given component, the direction of the electrical force is then that of the extended right hand.*

The electrical force in general forms a finite angle with the part of the conductor adjacent to the point in question. Naturally only the component falling in the direction of the conductor is effective. If in a given magnetic field the electrical force at a given point is to produce the greatest possible effect, then must this point move at right angles to the line of force, and the adjacent element of the conductor must stand at right angles to the plane, which is determined by the direction of the magnetic line of force and the direction of the

motion, or shortly: *The direction of the magnetic line of force, the direction of the motion, and the direction of the element, must be three directions perpendicular to each other.*

The electrical force is proportional to the strength of the magnetic field. If the magnetic field is produced by a coil, this is strengthened by the introduction of a soft iron rod to a very remarkable degree (even to thirty-three times). This is the reason why, in order to obtain as strong induced currents as possible, soft iron is placed inside the coil. The iron is usually in the form of a bundle of wire, the wires being annealed and insulated from each other. The annealing lessens the magnetic inertia, whilst the insulation prevents the induced currents that would arise in a massive iron core, and which would produce a detrimental effect.

APPENDIX

ELECTRICAL SYSTEMS OF MEASUREMENT

THE absolute system of measurement introduced by Gauss expresses all quantities on the basis of three fundamental units—the units of length, of mass, and of time. Gauss adopted the millimetre as unit of length; as unit of mass the mass of a milligramme, or, briefly stated, the milligramme; as unit of time the second. In England the units centimetre, gramme, and second were first introduced, and these units are now commonly adhered to. By gramme is understood here, naturally, not the weight but the mass of a cubic centimetre of chemically pure water at 4° centigrade.

If a quantity is measured by means of these units the result is expressed by a definite number. If however, for example, instead of the centimetre another length is chosen for the unit of length, this number becomes different. If k is the *reduction-factor*, *i.e.* that number which specifies how many of these new units of length there are in a centimetre, the number which we get for the quantity measured will be k^n times as great as before. The power n is named the *dimension* of the quantity considered in reference to the unit of length. If I measure, for example, a rectilinear distance in centimetres, and find it equal to 2 centimetres,

if I wish to express the same length in millimetres I find it equal to 2×10 or 20 millimetres. k is here 10, n equal to 1 ; therefore distance in respect to the unit of length is of the dimension 1. A surface which measures 2 square centimetres will, expressed in millimetres, amount to 2×10^2 or 200 square millimetres. Surface, therefore, has the dimension 2. If I have found a space equal to 2 cubic centimetres, then I at the same time obtain for this 2×10^3 or 2000 cubic millimetres ; space, therefore, has the dimension 3.

Conversely, if I would express the above quantities in terms of the metre, then I must remember that $\frac{1}{100}$ metre is equal to 1 centimetre. The reduction-factor is now $\frac{1}{100}$ or 10^{-2}. The distance of 2 centimetres will be equal to $2 \times (10^{-2})^1 = 0.02$ metre ; the area of 2 square centimetres is equal to $2 \times (10^{-2})^2 = 0.0002$ square metre ; the space containing 2 cubic centimetres will be equal to $2 \times (10^{-2})^3 = 0.000002$ cubic metre.

Exactly the same holds true in regard to mass and time. If we denote the units of length, mass, and time by l, m, t, and if we have to deal with a magnitude which is as regards length of the dimension $\frac{3}{2}$, as regards mass of the dimension $\frac{1}{2}$, and as regards time of the dimension -2, we express this by saying that the magnitude is of the dimension

$$[l^{\frac{3}{2}} m^{\frac{1}{2}} t^{-2}].$$

A distance is of the dimension $[l]$.

An area of the dimension $[l^2]$.

And a space (volume) of the dimension $[l^3]$.

A velocity is equal to a distance (length) divided by a time ; it is of the dimension

$$[lt^{-1}].$$

An acceleration is equal to a velocity divided by a time ; its dimension is $[lt^{-2}]$.

A force is equal to a mass × acceleration; its dimension is

$$[lmt^{-2}].$$

Work is equal force × distance passed over (length); its dimension is

$$[l^2mt^{-2}].$$

In electricity we have two systems of units which are both derived from the absolute system of measurement; the one is founded on the force which acts between two quantities of electricity, and is called the electrostatic system of measurement, the other on the force which is in action between two magnetic poles, and is called the electro-magnetic system of measurement.

1.—ELECTROSTATIC SYSTEM OF MEASUREMENT.

We have defined as the electrostatic unit of electricity that quantity of electricity which repels an equal quantity of electricity of the same name at the unit of distance with the unit of force. If two electrified points have charges of e and e' units, and if r is their distance, the force of their mutual action is equal $\frac{ee'}{r^2}$. This force has as force the dimension $[lmt^{-2}]$. The dimension of r^2 is $[l^2]$. If we denote the dimension of the electrostatic unit of electricity by $[x]$ we get the equation

$$\frac{[x][x]}{[l^2]} = [lmt^{-2}],$$

from which follows

$$[x] = [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}].$$

By electrical force we understand the force proceeding from certain electrical masses, and acting on the unit

of positive electricity. It is therefore equal to force÷quantity of electricity, and therefore its dimension is equal to

$$[lmt^{-2}] \div [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] = [l^{-1}m^{\frac{1}{2}}t^{-1}].$$

Potential and difference of potential are equal to work ÷quantity of electricity; their dimension is

$$[l^2mt^{-2}] \div [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}] = [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}].$$

Electrical density is quantity of electricity÷area; its dimension is

$$[l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] \div [l^2] = [l^{-\frac{3}{2}}m^{\frac{1}{2}}t^{-1}].$$

It is of the same dimension as the electrical force (*cf.* section 28, p. 85).

Capacity is equal to quantity of electricity÷potential; its dimension is

$$[l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] \div [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] = [l].$$

It is of the same dimension as a length (*cf.* the capacity of a sphere, section 40, p. 129).

The dielectric constant is equal to capacity÷capacity; its dimension is

$$[l] \div [l] = [l^0].$$

i.e. the dielectric constant is a constant numerical ratio.

The strength of the current is equal to electrical quantity÷time; its dimension is equal to

$$[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}] \div [t] = [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}].$$

Resistance is equal to difference of potential÷strength of current; its dimension is

$$[l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] \div [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}] = [l^{-\frac{1}{2}}t].$$

2.—Electro-magnetic System of Measurement.

That quantity which exerts the unit force on an equal quantity at the unit distance is considered as the unit of quantity of free magnetism.

If m and m' such units are situated at two points, and if r is the distance apart of these points, the force of the mutual action is equal

$$\frac{mm'}{r^2}.$$

It follows from this that the dimension of the unit of mass of free magnetism is equal to

$$[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}].$$

The magnetic moment is free magnetism $\times$ length; its dimension is

$$[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}] \times [l] = [l^{\frac{5}{2}}m^{\frac{1}{2}}t^{-1}].$$

If i is the strength of a current, measured electro-magnetically, and if it flows round an area f, it exerts the same magnetic effect at a distance as a short magnet fixed through the middle of f, perpendicular to the plane of the current and of magnetic moment fi. The product fi is therefore in its character a magnetic moment, or the strength of the current is equal to magnetic amount $\div$ area, and therefore its dimension is

$$[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-1}] \div [l^2] = [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}].$$

That current possesses the electro-magnetic unit of current strength which, flowing round the surface unity, acts at a distance like a short magnet of unit magnetic moment, fixed through the middle of the surface perpendicular to the plane of the current. The strength of the current is essentially electrical quantity $\div$ time; quantity of electricity is therefore equal to strength of current $\times$ time, and hence its dimension is equal to

$$[l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] \times [t] = [l^{\frac{1}{2}}m^{\frac{1}{2}}].$$

The electro-magnetic unit of electricity is that quantity of electricity which the unit current, measured electro-magnetically, transfers through a cross-section of the conductor in a second. The electro-magnetic unit of electricity is considerably greater than the electro-

static unit; the latter is contained in the former nearly $3 \times 10^{10}\left(\frac{\text{centimetre}}{\text{second}}\right)$ times.

Magnetic potential is equal to work÷quantity of free magnetism; its dimension is

$$[l^2mt^{-2}] \div [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] = [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}].$$

Electrical potential (potential = difference, electro-motive force) = work÷quantity of electricity; its dimension is equal to

$$[l^2mt^{-2}] \div [l^{\frac{1}{2}}m^{\frac{1}{2}}] = [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}].$$

Resistance is equal to potential difference÷strength of current; its dimension is equal to

$$[l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}] \div [l^{\frac{1}{2}}m^{\frac{1}{2}}t^{-1}] = [lt^{-1}].$$

It is of the same dimension as a velocity.

Capacity is equal to quantity of electricity÷potential; its dimension is

$$[l^{\frac{1}{2}}m^{\frac{1}{2}}] \div [l^{\frac{3}{2}}m^{\frac{1}{2}}t^{-2}] = [l^{-1}t^2].$$

If these units were adopted in practice the one set would be much too large, the other set much too small. Practical needs required other units, and this has led to a new system of measurement, the so-called practical system of measurement.

3.—The Practical System of Measurement.

The units of the practical system of measurement follow from the units of the electro-magnetic system of measurement, if we introduce for the unit of length instead of the centimetre the length of the earth's quadrant, *i.e.* 10^9 cm., and for the unit of mass instead of the gramme the mass of 10^{-11} gramme; and retain the second for the unit of time. In deducing the new units we must carefully bear in mind what we have said above about reduction-factors and "dimensions." The

reduction-factor for length is here 10^{-9} and the reduction-factor for mass is 10^{11}. Let us assume that we had in the electro-magnetic system of measurement obtained the number 1 throughout for all quantities introduced. If we now measure these quantities in accordance with the new system of measurement we naturally get other numbers, viz.—

For force . . . $10^{-9} \times 10^{11} = 10^{2} = 100$
For work . . . . $(10^{-9})^{2} \times 10^{11} = 10^{-7}$
For quantity of free magnetism $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10^{-8}$
For magnetic moment . $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10^{-17}$
For strength of current . $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10$
For quantity of electricity . $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10$
For magnetic potential . $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10$
For electrical potential . $(10^{-9})^{\frac{1}{2}} \times (10^{11})^{\frac{1}{2}} = 10^{-8}$
For resistance . . . . . 10^{-9}
For capacity . . . . . 10^{9}

We see from this that the present unit of force is 10^{2} or 100 times as small as the former. The present unit of work is 10^{7} as great as the former; it is named a *watt.*

The present unit of quantity of free magnetism is 10^{8} times greater than formerly; it is called a *weber.*

The present unit of magnetic moment is 10^{17} times greater than before.

The present unit of strength of current is 10^{-1} times as great, *i.e.* 10 times as small as before; it is called an *ampère.*

The present unit of electrical quantity is 10 times as small as the electro-magnetic unit; it is called a *coulomb.*

The present unit of magnetic potential is 10 times as small as before.

The present unit of electrical potential is 10^{8} times as great as before; it is named a *volt.*

The present unit of resistance is 10^9 times as great as before; it is named an *ohm.*

The present unit of capacity is 10^9 times as small as before; it is named a *farad.*

When in the course of investigations there occur quantities which are a million times as great or a million times as small as the quantities mentioned, the words "mega" and "micro" respectively are prefixed. "Mega" therefore denotes "1 million," "micro," on the other hand, "1 millionth." For example, a megohm is a resistance of 1,000,000 ohms; on the other hand a microfarad is a capacity of $\frac{1}{1,000,000}$ farad. The word "milli" is likewise used for "1 thousandth," 1 milliweber for example equals $\frac{1}{1000}$ weber.

In order to give an adequate notion of the units ohm, volt, ampère, the following may be stated:

A coulomb is that quantity of electricity which, flowing through a cross-section of the conductor in a second produces an ampère.

If in a closed galvanic element the electromotive force is equal to 1 volt and the resistance is equal to 1 ohm, the strength of the current is then equal to 1 ampère.

$$1 \text{ ampère} = \frac{1 \text{ volt}}{1 \text{ ohm}}.$$

1 watt = 1 ampère × 1 volt (Joule's law).

A condenser which has the capacity of 1 farad will, by a charge of 1 coulomb, be raised to the potential of 1 volt.

1 ohm = 1.06 Siemens unit. By Siemens unit is understood the resistance of a column of quicksilver 1 metre in length and 1 square millimetre in cross-section.

1 volt is about equal to 0.9 Daniell (= electromotive force of a Daniell's element).

1 ampère can deposit 0.001,118 grm. silver per second, or 4.025 grm. per hour; or 6.95 cubic centimetres of hydrogen, or 10.43 cubic centimetres of explosive gas (2 vols. hydrogen to 1 vol. oxygen) per minute, the gas being measured at 0° centigrade and under the pressure of 76 cm. mercury at 0°.

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